Compactness properties of perturbed sub-stochastic C 0 -semigroups on L 1 () with applications to discreteness and spectral gaps

Mustapha Mokhtar-Kharroubi

Compactness properties of perturbed sub-stochastic C_0-semigroups on $L^1(\mu)$ with applications to discreteness and spectral gaps

Mustapha Mokhtar-Kharroubi
Département de Mathématiques, CNRS-UMR 6623
Université de Franche-Comté
16 Route de Gray, 25030 Besançon, France.
E-mail: mmokhtar@univ-fcomte.fr

Abstract

We deal with positive C_0-semigroups $(U(t))_{t\geqslant 0}$ of contractions in $L^1(\Omega;\mathcal{A},\mu)$ with generator T where $(\Omega;\mathcal{A},\mu)$ is an abstract measure space and provide a systematic approach of compactness properties of perturbed C_0-semigroups $\left(e^{t(\text{“}T-V\text{”})}\right)_{t\geq 0}$ (or their generators) induced by singular potentials $V:(\Omega;\mu)\to\mathbb{R}_+$. More precise results are given in metric measure spaces (Ω,d,μ). This new construction is based on several ingredients: new a priori estimates peculiar to L^1-spaces, local weak compactness assumptions on unperturbed operators, “Dunford-Pettis” arguments and the assumption that the sublevel sets $\Omega_M := \{x; V(x)\leq M\}$ are “thin at infinity with respect to $(U(t))_{t\geqslant 0}$”. We show also how spectral gaps occur when the sublevel sets are not “thin at infinity”. This formalism combines intimately the kernel of $(U(t))_{t\geqslant 0}$ and the sublevel sets Ω_M. Indefinite potentials are also dealt with. Various applications to convolution semigroups, weighted Laplacians and Witten Laplacians on 1-forms are given.

Contents

1 Introduction

This work is an abridged (and improved) version of [49] and provides new functional analytic tools and results on perturbation theory and spectral analysis of substochastic C_0-semigroups in L^1 spaces and also various results of applied interest. Before outlining the content of this work, some related information in Hilbert space setting is worth mentioning. According to a classical result going back at least to K. Friedrichs [18], the spectrum of a Schrödinger operator in $L^2(\mathbb{R}^N)$

$$-\Delta \dotplus V \quad \text{(form-sum)}$$

is fully discrete (i.e. it consists of isolated eigenvalues with finite multiplicity) or equivalently $-\Delta \dotplus V$ has a compact resolvent for nonnegative potentials $V \in L^1_{loc}(\mathbb{R}^N)$ such that

$$\lim_{|x|\to\infty} V(x) = +\infty.$$

Of course, it is also known since a long time that this condition is not necessary since F. Rellich [61] already observed for example that for the potential

$$V(x_1,x_2) = x_1^2 x_2^2, \tag{1}$$

$-\Delta \dotplus V$ is still resolvent compact in $L^2(\mathbb{R}^2)$ even if $V(x_1,x_2)$ fails to go to $+\infty$ at infinity near the axes. Besides K. Friedrichs [18], the literature on discreteness of the spectrum of Schrödinger operators goes back to A.M. Molchanov [51] and is now considerable; we refer to the survey [65] and also to the more recent paper [39] for more developments. This literature deals with Schrödinger operators on more general non-compact Riemannian manifolds and provides optimal (i.e. necessary and sufficient) conditions of discreteness in terms of Wiener capacity of suitable sets. Such sharp results are not always of simple practical use, but sufficient or necessary conditions in terms of measures are also available. For instance, we note A.M. Molchanov's necessary condition of discreteness

$$\int_{B(x,r)} V(y)dy \to +\infty \ \text{ as } \ |x| \to \infty.$$

We note also that if for any $M > 0$ the sublevel set

$$\Omega_M := \{y; V(y) \leq M\}$$

is "thin at infinity" in the sense that for some $r > 0$

$$|B(x,r) \cap \Omega_M| \ \to 0 \ \text{ as } \ |x| \to \infty \tag{2}$$

($B(x,r)$ is the ball centered at x with radius r and $||$ refers to Lebesgue measure) then $-\Delta \dot{+} V$ in $L^2(\mathbb{R}^N)$ has a discrete spectrum, see [65] Corollary 10.2, p. 268.

In ([20] Lemma 5 and Remark 2) it is observed that the sublevel sets of a nonnegative function V are "thin at infinity" if and only if

$$\int_{B(x,r)} \frac{1}{1+V(y)} dy \to 0 \ \text{ as } \ |x| \to \infty \tag{3}$$

for some $r > 0$; the argument relies on the simple double inequality (for arbitrary $M > 0$)

$$\frac{1}{1+M} |B(x,r) \cap \Omega_M| \leq \int_{B(x,r)} \frac{1}{1+V(y)} dy$$

$$\int_{B(x,r)} \frac{1}{1+V(y)} dy \leq |B(x,r) \cap \Omega_M| + \frac{1}{1+M} |B(0,r)| \, .$$

One realizes then that the above sufficient criterion of discreteness coincides with the one already given in [6] under Assumption (3); one sees also that A.M. Molchanov's necessary condition follows from "thinness at infinity" of sublevel sets Ω_M since

$$\begin{aligned} |B(0,r)| &= |B(x,r)| = \int_{B(x,r)} \frac{\sqrt{1+V(y)}}{\sqrt{1+V(y)}} dy \\ &\leq (\int_{B(x,r)} \frac{1}{1+V(y)} dy)^{\frac{1}{2}} (\int_{B(x,r)} (1+V(y)) dy)^{\frac{1}{2}} \end{aligned}$$

and then

$$\int_{B(x,r)} V(y)dy \geqslant -|B(0,r)| + \frac{|B(0,r)|^2}{\int_{B(x,r)} \frac{1}{1+V(y)} dy};$$

it seems that this has not been noticed in the literature on the subject.

More recently, it was shown in [34] that $T - V$ is resolvent compact in $L^2(\mathbb{R}^N)$ when T is the relativistic α-stable operator

$$T = -(-\Delta + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} + m \tag{4}$$

provided that $\lim_{|x|\to\infty} V(x) = +\infty$. This result was extended in [72] (for sublevels sets Ω_M having finite measure only) to much more general symmetric Markov generators in $L^2(\Omega;\mu)$ satisfying the so-called *intrinsic super Poincaré inequality* and such that the Markov semigroup has a density with respect to μ. The proof given by the authors is however quite involved and combines various technical arguments; shortly after, a simpler proof was given in [69] and other developments, still for self-adjoint operators in Hilbert spaces, were also given in [20][35]. Even the finiteness assumption on the measure of the sublevels sets Ω_M has been dropped. For instance, we find in [69] that if T is a self-adjoint operator in $L^2(\Omega;\mu)$ such that $\{e^{tT}; t \geqslant 0\}$ is an *ultracontractive* C_0-semigroup in the sense that

$$e^{tT} \in \mathcal{L}(L^2(\Omega;\mu), L^\infty(\Omega;\mu)) \tag{5}$$

(for some $t > 0$) then $T - V$ is resolvent compact in $L^2(\Omega;\mu)$ provided that $V \in L^1_{loc}(\mathbb{R}^N)$ and $V \geqslant 0$ is such that its sublevels sets are r-polynomially thin (for some $r > 0$), i.e. for any $R > 0$

$$\int_{\Omega_M} |\Omega_M \cap B(x;R)|^r \mu(dx) < +\infty.$$

We note that in $\mathbb{R}^N$, r-polynomially thin set is necessarily thin at infinity in the sense (2) (see [20] Lemma 7).

There exists also an important literature on *Poincaré* (or spectral gap) inequality for Markov C_0-semigroups arising in Probability and Statistical Mechanics

$$var_\mu(f) := \int_\Omega f^2 d\mu - (\int_\Omega f d\mu)^2 \leq c(A^{\frac{1}{2}} f, A^{\frac{1}{2}} f), \quad f \in D(A^{\frac{1}{2}}),$$

(of interest e.g. for exponential trend to equilibrium) where (Ω, μ) is a probability space, A is a nonnegative self-adjoint operator in $L^2(\Omega,\mu)$, $1 \in D(A)$

and $A1 = 0$. Such an inequality is sometimes derived from Log Sobolev (or Gross) inequalities; see e.g. [24][63][26][73][3]. This notion of a spectral gap amounts to the fact that 0, the bottom of $\sigma(A)$, is an isolated simple eigenvalue; as such, it is meaningful in much more general (e.g. non hilbertian) contexts even if it cannot be formulated in terms of variance inequality. This inequality amounts to strict positivity of the bottom of the essential spectrum $\sigma_{ess}(A)$; we refer to [59][41] for the location of essential spectra of Schrödinger operators $-\Delta + V$ in $L^2(\mathbb{R}^N)$ when the sublevel sets of V are not "thin at infinity". We refer also to [13] for different related spectral problems. We point out that all the results above are *hilbertian* in nature; in particular neither L^1 compactness results nor spectral gap results in L^1 spaces can a priori be derived from this literature.

1.1 A new formalism in L^1 spaces

This work is intended to provide a new point of view on these spectral problems in abstract L^1 spaces. Let

$$(\Omega; \mathcal{A}, \mu)$$

denote a general measure space and let $(U(t))_{t\geqslant 0}$ be a positive C_0-semigroup of contractions (i.e. a substochastic C_0-semigroup) on $L^1(\Omega; \mathcal{A}, \mu)$ with generator T. In the sequel, for brevity, we will write $L^1(\Omega; \mu)$ or even $L^1(\Omega)$ unstead of $L^1(\Omega; \mathcal{A}, \mu)$. We denote by

$$V : \Omega \to \mathbb{R}_+,$$

a nonnegative (or more generally bounded from below) finite almost everywhere measurable function, i.e.

$$0 \leq V(x) < +\infty \text{ a.e.} \tag{6}$$

Let $V_n := V \wedge n$ and $\left(e^{t(T-V_n)}\right)_{t\geqslant 0}$ be the C_0-semigroup generated by $T - V_n$. It is elementary to see that

$$e^{t(T-V_{n+1})} f \leq e^{t(T-V_n)} f \quad \forall f \in L^1_+(\Omega; \mu)$$

so that a monotone convergence in $L^1(\Omega; \mu)$

$$U_V(t) f := \lim_{n\to+\infty} e^{t(T-V_n)} f \tag{7}$$

defines a semigroup $(U_V(t))_{t\geqslant 0}$. This semigroup is a priori strongly continuous for $t > 0$ only (see e.g. [2]). We say that

$$V \text{ is } \textit{admissible} \text{ for } \{U(t); t \geqslant 0\} \tag{8}$$

if $(U_V(t))_{t\geqslant 0}$ is a C_0-semigroup, i.e. is strongly continuous at $t=0$. In such a case, T_V, the generator of $(U_V(t))_{t\geqslant 0}$, is an *extension* of

$$T - V : D(T) \cap D(V) \to L^1(\Omega;\mu).$$

Note that if

$$D(T) \cap D(V) \text{ is dense in } L^1(\Omega;\mu)$$

then V is admissible for $\{U(t); t \geqslant 0\}$, see [71] Proposition 2.9. (The above considerations hold in all L^p spaces, see [71]. Actually, this construction extends in case $(U(t))_{t\geqslant 0}$ is *not* positive but is dominated by a positive contraction C_0-semigroup and also to *complex* potentials V, see [37].)

In this paper, we deal with spectral theory of perturbed C_0-semigroups $(U_V(t))_{t\geqslant 0}$ or perturbed generators T_V. More precisely, we are concerned with resolvent compactness of T_V and, more generally, with existence of *spectral gaps* for perturbed generators, i.e.

$$s_{ess}(T_V) < s(T_V) \tag{9}$$

where

$$s(T_V) := \sup\{\operatorname{Re}\lambda; \lambda \in \sigma(T_V)\}$$

is the spectral bound of T_V and

$$s_{ess}(T_V) := \sup\{\operatorname{Re}\lambda; \lambda \in \sigma_{ess}(T_V)\}$$

is the essential spectral bound of T_V, (σ_{ess} refers to essential spectrum). Note that $s(T_V) \in \sigma(T_V)$ and $s(T_V)$ *coincides* with the type ω of $(U_V(t))_{t\geqslant 0}$ from classical theory of positive C_0-semigroups on L^p spaces (see [52][74]). Note also that (9) implies that

$$\sigma(T_V) \cap \{\lambda;\ \operatorname{Re}\lambda > s_{ess}(T_V)\}$$

consists of a nonempty set of isolated eigenvalues with finite algebraic multiplicities.

We study also the compactness of the perturbed C_0-semigroup $(U_V(t))_{t\geqslant 0}$ and, more generally, its *essential compactness* i.e.

$$r_{ess}(U_V(t)) < r_\sigma(U_V(t)) \tag{10}$$

where $r_\sigma(U_V(t))$ is the spectral radius of $U_V(t)$ ($r_\sigma(U_V(t)) = e^{\omega t}$) and

$$r_{ess}(U_V(t)) := \sup\{|\mu|; \mu \in \sigma_{ess}(U_V(t))\}$$

is the essential spectral radius of $U_V(t)$.

Note that we can attach to $(U_V(t))_{t\geqslant 0}$ an *essential* type $\omega_{ess} \in [-\infty, s(T_V)]$ such that

$$r_{ess}(U_V(t)) = e^{\omega_{ess} t} \quad (t \geq 0),$$

(see e.g. [52] p. 73-74). We say that $(U_V(t))_{t\geqslant 0}$ has a *spectral gap* if (10) is satisfied or equivalently if

$$\omega_{ess} < s(T_V).$$

Similarly, (10) implies that

$$\sigma(U_V(t)) \cap \{\beta;\ |\beta| > r_{ess}(U_V(t))\}$$

consists of a nonempty set of isolated eigenvalues with finite algebraic multiplicities; this in turn implies that

$$\sigma(T_V) \cap \{\lambda;\ \operatorname{Re}\lambda > \omega_{ess}\}$$

consists of a nonempty set of isolated eigenvalues with finite algebraic multiplicities (see e.g. [52]) and consequently

$$s_{ess}(T_V) \leq \omega_{ess}.$$

Thus the existence of a spectral gap for $(U_V(t))_{t\geqslant 0}$ implies that T_V has a spectral gap while the converse statement is not true in general. Indeed, in practice, unless we know that $(U_V(t))_{\ t\geqslant 0}$ is *operator norm continuous*, i.e.

$$(0, +\infty) \ni t \to U_V(t) \in \mathcal{L}(L^1(\Omega; \mu))$$

is continuous in operator norm, a priori we do not have a spectral mapping theorem for $(U_V(t))_{t\geqslant 0}$ so that, in general, its spectral properties cannot be completely inferred from the knowledge of $\sigma(T_V)$.

Here the essential spectrum $\sigma_{ess}(O)$ of a closed linear operator O on a Banach space X is the complement of its Fredholm domain. It is known that if $O \in \mathcal{L}(X)$ then

$$\sigma_{ess}(O + S) = \sigma_{ess}(O)$$

for any *strictly singular* operator S (see e.g. [36], Proposition 2.c.10, p. 79) and consequently

$$r_{ess}(O + S) = r_{ess}(O).$$

(We point out that there are several non equivalent concepts of essential spectra but, for bounded operators, the corresponding essential spectral radius is the *same* for all them, see [17] Corollary 4.11, p. 44.) It is known also that in L^1-*spaces* the class of strictly singular operators is nothing but the class of *weakly compact* operators, see [58]. The use of weak compactness turns out to be the right tool for spectral theory in L^1 spaces. Indeed, most of our proofs rely on weak compactness arguments.

We are particularly interested in case the unperturbed C_0-semigroup $(U(t))_{t\geqslant 0}$ is neither compact nor essentially compact. A new and systematic approach of compactness or essential compactness properties of perturbed C_0-semigroups $(U_V(t))_{t\geqslant 0}$ (*induced* by singular potentials V) is provided. While most of the known literature on full discretenes or spectral gaps is concerned with hilbertian results and quite often by self-adjoint semigroups, we provide here a new point of view relying on a new circle of ideas peculiar to L^1-spaces without any connection with self-adjointness. In our general context, the relevant technical tools we need will be different depending on whether we deal with T_V or $(U_V(t))_{t\geqslant 0}$. Thus, in our study of spectral properties of perturbed generators T_V, we take advantage of the quite unsuspected fact (in comparison to L^2-space setting) that V is *always* T_V-bounded in L^1 spaces [53][71]. On the other hand, to study spectral properties of perturbed C_0-semigroups $(U_V(t))_{t\geqslant 0}$, we provide *two different strategies.* The first strategy consists in assuming that $(U(t))_{t\geqslant 0}$ is operator norm continuous, (i.e.

$$(0, +\infty) \ni t \to U(t) \in \mathcal{L}(L^1(\Omega; \mu))$$

is continuous in operator norm), in showing the operator norm continuity of the perturbed C_0-semigroup $(U_V(t))_{t\geqslant 0}$ and in taking advantage of spectral properties of T_V and *"spectral mapping tools"* for operator norm continuous C_0-semigroups. In a second (direct) strategy, we show a *"weak type"* estimate for $t > 0$

$$\int_{\{V>M\}} (U_V(t)f)\mu(dx) \leq \frac{c_t \|f\|}{M}, \forall\ f \in L^1_+(\Omega; \mu),\ \forall\ M > 0 \tag{11}$$

under the assumption

$$c_t := \lim_{\varepsilon\to 0_+}\inf \left\| \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega;\mu)} < +\infty,\ (t > 0) \tag{12}$$

where $U_V^*(t)$ is the dual operator of $U_V(t)$. Actually, the contractivity of $(U_V(t))_{t\geqslant 0}$ shows that (12) is equivalent to

$$\lim_{\varepsilon\to 0_+}\inf \left\| \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega;\mu)} < +\infty,\ (t \in (0, \delta)) \tag{13}$$

for some small $\delta > 0$.

The weak type estimate (11) provides us with an alternative approach of compactness or essential compactness of perturbed C_0-semigroups $(U_V(t))_{t\geqslant 0}$ when $(U(t))_{t\geqslant 0}$ is not a priori operator norm continuous. A sufficient condition for (12) to hold is

$$\lim_{\varepsilon\to 0_+} \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \text{ exists in weak star topology}$$

i.e.

$$U_V^*(t)1 \in D((T_V)^*), \quad \forall t > 0 \tag{14}$$

($(T_V)^*$, the dual of T_V, is the weak star generator of $(U_V^*(t))_{t\geqslant 0}$) or equivalently

$$\forall f \in L^1(\Omega;\mu), (0,+\infty) \ni t \to \int U_V(t)f \ \text{ is differentiable.} \tag{15}$$

If

$$(0,+\infty) \ni t \to U_V^*(t)1 \in L^\infty(\Omega) \text{ is continuous} \tag{16}$$

(e.g. if $1 \in \overline{D((T_V)^*)}$) then (12) and (14) turn out to be *equivalent.*

We point out that (15) is much weaker than a differentiability assumption on $(U_V(t))_{t\geqslant 0}$.

A peculiarity of Assumption (12) is that it concerns the *dual* perturbed C_0-semigroup $(U_V^*(t))_{t\geqslant 0}$ which is not a priori a "given object" in contrast to $(U(t))_{t\geqslant 0}$ and V. The good news is that (12) is always satisfied if

$$(U(t))_{t\geqslant 0} \text{ is holomorphic} \tag{17}$$

because $(U_V(t))_{t\geqslant 0}$ is then holomorphic too [2][30]. On the other hand, it is an open problem (even for bounded V) to decide whether a differentiability of $(U(t))_{t\geqslant 0}$ can be inherited by $(U_V(t))_{t\geqslant 0}$ regardless of V (see e.g. [62]). Note that a sufficient condition of (immediate) differentiability of a contraction C_0-semigroup $(S(t))_{t\geqslant 0}$ with generator G is

$$\exists \omega > 0, \lim_{|s|\to\infty} \ln|s| \left\|(\omega + is - G)^{-1}\right\| = 0, \tag{18}$$

see ([57] Corollary 4.10, p. 58). We denote by $\mathcal{P}$ the class of C_0-semigroups of contractions with generators satisfying (18) and show that if $(U(t))_{t\geqslant 0}$ belongs to $\mathcal{P}$ and if V belongs to its generalized Kato-class potentials, i.e.

$$V \text{ is } T\text{-bounded and } \lim_{\lambda\to+\infty} r_\sigma\left[V(\lambda - T)^{-1}\right] < 1 \tag{19}$$

then $(U_V(t))_{t\geqslant 0}$ belongs also to $\mathcal{P}$. Thus (12) is also satisfied for the class-$\mathcal{P}$ differentiable C_0-semigroups $(U(t))_{t\geqslant 0}$ and their generalized Kato-class potentials V.

We point out that (12) could hold for C_0-semigroups $(U_V(t))_{t\geqslant 0}$, which are *not even* operator norm continuous (see Proposition 6 and Proposition 7 below). *A further understanding of the validity of (12) outside the class of operator norm continuous C_0-semigroups is thus a very interesting open problem.*

Actually under (12) we obtain the following result which is more general than the weak type estimate (11):

$$U_V(t) \in \mathcal{L}(L^1(\Omega;\mu); D(V)) \quad (\forall t > 0) \tag{20}$$

where $D(V)$ is the domain of the multiplication operator by V endowed with the graph norm. The smoothing effect (20) has been obtained with M. Brassart and did not appear in the initial version [49] of this paper where the weak type estimate (11) only is obtained for *almost all* $t > 0$ under the additional assumption that $L^1(\Omega;\mathcal{A},\mu)$ is *separable.*

The fact that V is T_V-bounded, the weak type estimate (11) combined to local weak compactness assumptions on unperturbed operators, to properties of sublevel sets

$$\Omega_M := \{y; V(y) \leq M\},$$

more precisely their "size at infinity with respect to unperturbed operators" (see the definition below), and to "Dunford-Pettis" arguments, play an important part in our formalism and provide us with new relevant tools in spectral theory of perturbed sub-stochastic C_0-semigroups. Our *local* L^1 weak compactness assumptions on unperturbed operators are very weak ones and are trivially satisfied by most examples occuring in the literature. We provide thus a pure L^1 theory on full discretenes or spectral gaps of perturbed substochastic C_0-semigroups.

For sub-Markov C_0-semigroups $(U(t))_{t\geqslant 0}$ (i.e. which act in all L^p spaces as positive contraction semigroups), the L^1 spectral picture extends to L^p spaces, providing us e.g. with hilbertian results, (while converse statements are not true in general, see [11] Section 4.3). However, our aim here is rather to build and explore an L^1 spectral theory for its own sake; as far as we know, this program is undertaken here for the first time.

Our approach of the subject suits C_0-semigroups exhibiting integral kernels. The existence of such kernels is a consequence of our local weak-compactness assumptions, (see Remark 23 below). We have in mind various kinds of transition kernels which appear in the literature on Markov processes in metric spaces. For instance, the Heat kernel associated to the Laplace Beltrami operator on non-compact complete Riemannian manifolds (Ω, d, μ) of dimension n (d is the geodesic distance and μ is the Riemannian volume) with Ricci curvature bounded below and having the so-called "bounded geometry" (see [11] p. 172) satisfies a Gaussian estimate for each $t > 0$

$$p_t(x, y) \leq C_t^1 \exp(-\frac{d(x, y)^2}{C_t^2}), \tag{21}$$

see e.g. [11][22]. However, Brownian motions on some fractal spaces lead to transition kernels with sub-Gaussian estimates

$$p_t(x, y) \leq \frac{C}{t^{\frac{\alpha}{\beta}}} \exp(-(\frac{d^{\beta}(x, y)}{Ct})^{\frac{1}{\beta-1}}) \tag{22}$$

where $\alpha > 0$ is the Hausdorff dimension and $\beta > 2$ is "a walk dimension", see e.g. [5]. On the other hand, the study of kernel estimates for non local Dirichlet forms, in connection with Markov processes with jumps, developped also in the last decades and typical kernel estimates of jump Markov C_0-semigroups are polynomial

$$p_t(x, y) \leq \frac{C}{t^{\frac{\alpha}{\beta}}}(1 + \frac{d(x, y)}{t^{\frac{1}{\beta}}})^{-(\alpha+\beta)}, \tag{23}$$

see e.g. [28].

1.2 Main results

Before outlining our main results, we mention first a useful *abreviation* used in all the paper in order to avoid cumbersome notations: for any linear operator $O \in \mathcal{L}(L^1(\Omega; \mu))$ and for any measurable subset $\Xi \subset \Omega$, the (abuse of) notation

$$O : L^1(\Omega; \mu) \to L^1(\Xi; \mu)$$

refers to the operator

$$L^1(\Omega; \mu) \ni f \to [Of]_{|\Xi} \in L^1(\Xi; \mu),$$

where $[Of]_{|\Xi}$ is the *restriction* of Of to the subset Ξ.

Section 2 is devoted to various technical results. We show the weak type estimate (11) from Assumption (12). Actually, we show first a more general result, the smoothing effect (20); its proof consists actually in pushing further the proof of the weak type estimate given in [49]. We show also how

(15) implies (12) and why they are equivalent if (16) is satisfied. Besides the class of holomorphic C_0-semigroups $(U(t))_{t\geqslant 0}$, we show how (11) is satisfied for class-$\mathcal{P}$ differentiable C_0-semigroups $(U(t))_{t\geqslant 0}$ and their generalized Kato class potentials V.

We show also the stability estimate for arbitrary $C>0$

$$\sup_{t\leq C}\left\|e^{t(T-V_n)}f-U_V(t)f\right\|\leq e^C\left\|[V-V_n]\,(1-T_V)^{-1}f\right\|,\ \forall f\in L^1_+(\Omega;\mu)$$

where $V_n := V\wedge n$. Note that $\left\{[V-V_n]\,(1-T_V)^{-1}\right\}_n$ is a sequence of bounded operators going strongly to zero as $n\to+\infty$. This estimate implies that $(U_V(t))_{t\geqslant 0}$ is operator norm continuous provided that $(U(t))_{t\geqslant 0}$ is operator norm continuous and

$$\left\|[V-V_n]\,(1-T_V)^{-1}\right\|_{\mathcal{L}(L^1(\Omega;\mu))}\to 0 \text{ as } n\to+\infty. \tag{24}$$

Finally, we show how weighted translation C_0-semigroups $(U_V(t))_{t\geqslant 0}$ on $L^1(\mathbb{R})$ satisfy (12) although they are *not* operator norm continuous.

Section 3 contains our main compactness theorems for general measure spaces $(\Omega;\mathcal{A},\mu)$. We show that T_V is resolvent compact provided that

$$(\lambda-T)^{-1}:L^1(\Omega;\mu)\to L^1(\Omega_M;\mu) \text{ is weakly compact} \tag{25}$$

where

$$\Omega_M:=\{y;V(y)\leq M\}$$

are the sublevel sets of V.

If $(U_V(t))_{t\geqslant 0}$ is operator norm continuous then (25) implies the stronger result that the perturbed C_0-semigroup $(U_V(t))_{t\geqslant 0}$ is compact on $L^1(\Omega;\mu)$. We can also avoid the operator norm continuity assumption. Indeed, if (12) is satisfied then we show that $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup on $L^1(\Omega;\mu)$ provided that

$$U(t):L^1(\Omega;\mu)\to L^1(\Omega_M;\mu)\ \text{ is weakly compact } (t>0,\ M>0). \tag{26}$$

Before proceeding further the general theory, we devote Section 4 to a specific class of C_0-semigroups, the so-called convolution semigroups (related to Lévy processes) on euclidean spaces because of their great applied interest. We show first a preliminary technical result:

Let $h\in L^1(\mathbb{R}^N)$ and let

$$H:L^1(\mathbb{R}^N)\ni\varphi\to\int_{\mathbb{R}^N}h(x-y)\varphi(y)dy\in L^1(\mathbb{R}^N)$$

be the corresponding convolution operator on $L^1(\mathbb{R}^N)$. For any Borel set $\Xi \subset \mathbb{R}^N$, we *characterize* the compactness of

$$H : L^1(\mathbb{R}^N) \to L^1(\Xi);$$

in particular, a sufficient condition for this to happen is that Ξ be "thin at infinity" in the sense (2). This allows us to deal with convolution C_0-semigroups $(U(t))_{t\geqslant 0}$

$$U(t) : f \in L^1(\mathbb{R}^N) \to \int f(x-y)m_t(dy) \in L^1(\mathbb{R}^N)$$

where $\{m_t\}_{t\geq 0}$ are Radon sub-probability measures on $\mathbb{R}^N$ such that $m_0 = \delta_0$ (the Dirac measure at zero), $m_t * m_s = m_{t+s}$ and $m_t \to m_0$ vaguely as $t \to 0_+$. The sub-probability measures $\{m_t\}_{t\geqslant 0}$ are characterized by

$$\widehat{m_t}(\zeta) := (2\pi)^{-\frac{N}{2}} \int e^{-i\zeta.x} m_t(dx) = (2\pi)^{-\frac{N}{2}} e^{-tF(\zeta)},\ \zeta \in \mathbb{R}^N \tag{27}$$

where $F(\zeta)$ is the so-called characteristic exponent; (see e.g. [31] Chapter 3). The resolvent of the generator T is also a convolution with a measure m^λ

$$(\lambda - T)^{-1} f = \int f(x-y)m^\lambda(dy)$$

where

$$\widehat{m^\lambda}(\zeta) = \int_0^{+\infty} e^{-\lambda t}\widehat{m_t}(\zeta)dt = \frac{1}{\lambda + F(\zeta)}.$$

Thus, if $m^\lambda(dy)$ is a function (i.e. is absolutely continuous with respect to Lebesgue measure) then T_V has a compact resolvent provided that the sublevel sets Ω_M are "thin at infinity" in the sense (2). Similarly, if $m_t(dy)$ are functions ($t > 0$) then $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup provided that (12) is satisfied and the sublevel sets Ω_M are "thin at infinity"; in addition, this property is shown to be stable by subordination. For instance, this covers all C_0-semigroups subordinated to the heat semigroup, e.g. the symmetric stable semigroup of order 2α, the geometric α-stable semigroup, the relativistic α-stable semigroup etc.

Section 5 complements Section 3 in the context of L^1 spaces over *separable metric measure spaces*, i.e. separable metric spaces (Ω, d) endowed with a Borel measure μ which is finite on bounded Borel subsets of Ω. This framework is motivated by Markov processes in metric spaces, (see e.g. [23]). The existence of a metric d allows to complement the main compactness results of Section 3, in particular to understand further the key conditions (25)(26) in terms of *"thinness at infinity"* of sublevel sets Ω_M. We restrict ourselves to the relevant case

$$\mu(\Omega) = +\infty.$$

We show that if (12) is satisfied and if $U(t)$ is such that

$$U(t) : L^1(\Omega; \mu) \to L^1(\Xi; \mu)$$

is weakly compact for any bounded Borel set $\Xi \subset \Omega$ then $(U_V(t))_{t \geqslant 0}$ is a compact C_0-semigroup in $L^1(\Omega; \mu)$ provided that for some $x_0 \in \Omega$

$$\lim_{C \to +\infty} \sup_{y \in \Omega} \int_{\{x \in \Omega_M;\ d(x,x_0) \geqslant C\}} p_t(x, y) \mu(dx) = 0 \tag{28}$$

where $p_t(x, y)$ is the kernel of $U(t)$.

We express (28) by saying that the sublevel sets Ω_M are *"thin at infinity with respect to* $(U(t))_{t \geqslant 0}$*"*. In particular, if

$$v(r) := \sup_{x \in \Omega} \mu(B(x, r)) < \infty \quad (r \geqslant 0)$$

and if $p_t(.,.)$ satisfies an estimate of the form

$$p_t(x, y) \leq f_t(d(x, y))$$

where

$$f_t : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is non increasing}$$

and such that (for large r) the function

$$r \to f_t(r) v(r + 1)$$

is nonincreasing and integrable at infinity then the sublevel sets Ω_M are "thin at infinity with respect to $(U(t))_{t \geqslant 0}$" if they are *"thin at infinity"* in the sense there exists a point $\overline{y} \in \Omega$ such that for any $R > 0$

$$\mu\{\Omega_M \cap B(y; R)\} \to 0 \ \text{ as } d(y, \overline{y}) \to +\infty.$$

These results apply e.g. to kernels with estimates of the form (21) (22) or (23) under an appropriate condition on the volume growth

$$r \to v(r)$$

(in order to meet the conditions on $r \to f_t(r)v(r+1)$), see Remark 34 below.

In Section 6 (which continues Section 5), we show how *spectral gaps occur* when the sublevel sets Ω_M are *not* "thin at infinity with respect to $(U(t))_{t\geqslant 0}$", more precisely, when (28) is *not* satisfied. Indeed, we show that if (12) is satisfied, if

$$U(t) : L^1(\Omega) \to L^1(\Xi)$$

is weakly compact for any bounded Borel set Ξ and if the kernel $p_t(x,y)$ of $U(t)$ satisfies the estimate

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < e^{s(T_V)t} \tag{29}$$

(for some $x_0 \in \Omega$) then the perturbed C_0-semigroup $(U_V(t))_{t\geqslant 0}$ exhibits a spectral gap (i.e. is essentially compact); more precisely, we show that

$$\omega_{ess} \leq \inf_{t>0} \frac{1}{t} \ln \left(\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) \right)$$

where ω_{ess} is the essential type of $(U_V(t))_{t\geqslant 0}$. To get some insight into (29), it is useful to have in mind that $s(T_V)$ is the type of $(U_V(t))_{t\geqslant 0}$ and that

$$e^{s(T_V)t} = r_\sigma(U_V(t)) \leq \|U_V(t)\|_{\mathcal{L}(L^1(\Omega))} \leq \|U(t)\|_{\mathcal{L}(L^1(\Omega))} = \sup_{y\in\Omega} \int_\Omega p_t(x,y)\mu(dx).$$

We also study spectral gaps for generators T_V. Indeed, we show that if the kernel $G_1(x,y)$ of $(1-T)^{-1}$ satisfies the estimate

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} G_1(x,y)\mu(dx) < \frac{1}{1-s(T_V)} \tag{30}$$

(for some $x_0 \in \Omega$) then the perturbed generator T_V exhibits a spectral gap; more precisely

$$s(T_V) - s_{ess}(T_V) \geq \frac{\eta((1-s(T_V))}{r_{ess}\left[(1-T_V)^{-1}\right]}$$

where η is the difference between the right and left hand sides of (30). Similiarly, we gain some insight into (30) by noting that

$$\sup_{y\in\Omega} \int G_1(x,y)\mu(dx) = \left\|(1-T)^{-1}\right\| \geq r_\sigma\left((1-T)^{-1}\right) = \frac{1}{1-s(T)} \geq \frac{1}{1-s(T_V)}.$$

Thus, under (30), $\sigma(T_V) \cap \{\lambda;\ \operatorname{Re}\lambda > s_{ess}(T_V)\}$ consists of a nonempty set of isolated eigenvalues with finite algebraic multiplicities. This spectral picture does not prevent a priori the existence of sequences of isolated eigenvalues of T_V with imaginary parts going to infinity. If additionally $(U_V(t))_{t\geqslant 0}$ is operator norm continuous then we get the much stronger conclusion that this C_0-semigroup has a spectral gap, i.e. is essentially compact.

We point out that the main results above extend more generally in case $(U(t))_{t\geqslant 0}$ is *not* positive but is dominated by a positive contraction C_0-semigroup $\left(\widetilde{U}(t)\right)_{t\geqslant 0}$ i.e.

$$|U(t)f| \leq \widetilde{U}(t)(t)\,|f|\,, \quad f \in L^1(\Omega)$$

(note that any contraction C_0-semigroup in L^1 space admits a *modulus*, i.e. a minimal dominating positive contraction C_0-semigroup [33]) and to *complex* potentials V provided that $\operatorname{Re} V$ is nonnegative and admissible with respect to $\left(\widetilde{U}(t)\right)_{t\geq 0}$ and $|\operatorname{Im} V|$ is regular with respect to $\left(\widetilde{U}(t)\right)_{t\geq 0}$, (see [37] for the definition of regularity). Indeed, in this case

$$|U_V(t)f| \leq \widetilde{U}_{\operatorname{Re} V}(t)\,|f|\,, \quad f \in L^1(\Omega)$$

(see [37] Proposition 1. 20 (a)) and then the role played here by $(U(t))_{t\geqslant 0}$ and V should be played respectively by $\left(\widetilde{U}(t)\right)_{t\geq 0}$ and $\operatorname{Re} V$ because weak compactness properties are stable by domination. We do not try to elaborate on these points here.

In Section 7, we deal with some *weighted* Laplacians on euclidean spaces (see e.g. [11][22][27][19]); we revisit and complement several L^2 compactness results given in [27] in connection with Fokker-Planck operators. Indeed, let $\mu(dx) = e^{-\Phi(x)}dx$ be a measure on $\mathbb{R}^N$ and let $-\triangle^\mu$ be the positive self-adjoint operator on $L^2(\mathbb{R}^N;\mu(dx))$ associated to the Dirichlet form

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2\,\mu(dx).$$

Then $\triangle^\mu$ is is unitarily equivalent to a Schrödinger operator on $L^2(\mathbb{R}^N;dx)$

$$\triangle - \left(\frac{1}{4}\,|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi\right).$$

Then, assuming that

$$V := \frac{1}{4}\,|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi$$

is bounded from below, we give several new compactness results on Schrödinger C_0-semigroups *in L^1 spaces* for various classes of potentials Φ arising in the literature (related to the fact that the sublevel sets of V are thin at infinity). More generally, we deal also with spectral gaps (when the sublevel sets of V are *not* thin at infinity); in particular, if $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi \geqslant 0$ and $e^{-\Phi} \in L^1(\mathbb{R}^N; dx)$ then the existence of a *spectral gap* for $\triangle^\mu$ is guaranteed under the condition

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{\{x\in\Omega_M;\ |x|\geqslant C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-\frac{|x-y|^2}{4t})dx < 1$$

where Ω_M are the sublevel sets of $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi$; (while $\triangle^\mu$ need *not* be resolvent compact). Thus, this condition provides us with a sufficient criterion for a probability measure $Z^{-1}e^{-\Phi(x)}dx$ on $\mathbb{R}^N$ to satisfy the *Poincaré inequality.*

In Section 8, we deal with Witten Laplacians, i.e. Hodge Laplacians on *weighted* forms (i.e. forms with coefficients in $L^2(\mathbb{R}^N; e^{-\Phi(x)}dx)$); see e.g. [70][32] and [26] Chapter 2. The Witten Laplacian on 0-forms is unitarily equivalent to

$$\triangle_\Phi^{(0)} = \triangle^{(0)} + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi$$

in $L^2(\mathbb{R}^N; dx)$ (where $\triangle^{(0)} = -\triangle$) while the Witten Laplacian on 1-forms is unitarily equivalent to

$$\triangle_\Phi^{(1)} = \triangle_\Phi^{(0)} \otimes Id + Hess\Phi$$

in $\left(L^2(\mathbb{R}^N; dx)\right)^N$ (1-forms are identified to their coefficients); both Laplacians are nonnegative and the lower spectral bound of $\triangle_\Phi^{(0)}$ is equal to *zero* when $e^{-\Phi(x)}dx$ is a probability measure. The interest of Witten Laplacians in Statistical Mechanics stems in particular from the beautiful Helffer- Sjöstrand's covariance formula

$$\int (f(x) - \langle f\rangle)(g(x) - \langle g\rangle)e^{-\Phi(x)}dx = \int \left((\triangle_\Phi^{(1)})^{-1}df, dg\right) e^{-\Phi(x)}dx, \quad (31)$$

where $\langle f\rangle = \int f(x)e^{-\Phi(x)}dx$ (see [70][32] and [26] Chapter 2). The invertibility of $\triangle_\Phi^{(1)}$ is of course a key point, (see [32] for the details). We show the existence of spectral connections between $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$: By combining L^1 results and hilbertian tools (Glazman's Lemma) we show here that if Φ is convex (no strict convexity is needed) then the *essential* lower spectral bound of $\triangle_\Phi^{(0)}$ is less than or equal to that of $\triangle_\Phi^{(1)}$; in particular $\triangle_\Phi^{(1)}$ is resolvent compact if $\triangle_\Phi^{(0)}$ is. We show also, still for convex Φ, that if $\triangle_\Phi^{(0)}$

has spectral gap and if the lowest eigenvalue λ_Φ of $Hess\Phi$ is not identically zero then the spectral lower bound of $\triangle_\Phi^{(1)}$ is strictly larger than that of $\triangle_\Phi^{(0)}$ and consequently $\triangle_\Phi^{(1)}$ is invertible if $e^{-\Phi(x)}dx$ is a probability measure. In such a case, (31) is thus meaningful while Brascamp-Lieb's inequality

$$\int (f(x) - \langle f\rangle)(g(x) - \langle g\rangle)e^{-\Phi(x)}dx \leq \left((Hess\Phi)^{-1}df, dg\right)$$

needs Φ to be uniformly strictly convex (see [32]). We can also remove the convexity assumption and study the existence of a spectral gap for $\triangle_\Phi^{(1)}$ in terms of the heat kernel and the sublevel sets of

$$\frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi.$$

In Section 9, we come back to the general theory in $L^1(\Omega; \mathcal{A}, \mu)$ and consider *indefinite* potentials

$$V = V_+ - V_-$$

(which are not a priori bounded from below); we regard "$T - (V_+ - V_-)$" as perturbed operators

$$T_{V_+} + V_-$$

provided that V_- is T_{V_+}-bounded and belongs to the generalized Kato class of $(e^{tT_{V_+}})_{t\geq 0}$. This second perturbation theory uses different ideas inspired by transport theory [42][47][48]. In particular, we show how the compactness or essential compactness properties of $(e^{tT_{V_+}})_{t\geq 0}$ are *inherited* by $(e^{t(T_{V_+}+V_-)})_{t\geq 0}$. Finally, for sub-Markov C_0-semigroups $(U(t))_{t\geqslant 0}$, we show how these results extend to L^p spaces.

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2 Preliminary results

In this section (and in the following one), $(\Omega; \mathcal{A}, \mu)$ denotes a general measure space and $(U(t))_{t\geqslant 0}$ is a sub-stochastic C_0-semigroup on $L^1(\Omega; \mathcal{A}, \mu)$ with generator T. We denote by $(U_V(t))_{t\geqslant 0}$ the sub-stochastic C_0-semigroup on $L^1(\Omega; \mathcal{A}, \mu)$ defined in the Introduction (see (7)) where V is a *nonnegative* potential satisfying (6) and admissible for $(U(t))_{t\geqslant 0}$. This section is devoted to several technical results. We start with the following known result peculiar to L^1-spaces [53][71]; for reader's convenience, we recall briefly its proof (as given in [71] Lemma 4.1) in a slightly different form.

Lemma 1 *Let V satisfy (6)(8). Then $D(T_V) \subset D(V)$ and V is T_V-bounded.*

Proof. For a bounded potential W and $f \in D(T) \cap L^1_+(\Omega;\mu)$ we have for any real λ

$$\begin{aligned}
\frac{d}{dt}\left\|e^{-\lambda t}U_W(t)f\right\| &= \frac{d}{dt}\int e^{-\lambda t}U_W(t)f\ d\mu = \int \frac{d}{dt}\left[e^{-\lambda t}U_W(t)f\ \right] d\mu \\
&= \int (T-\lambda-W)\left[e^{-\lambda t}U_W(t)f\right]\ d\mu \\
&= \int (T-\lambda)\left[e^{-\lambda t}U_W(t)f\right]\ d\mu - \int W\left[e^{-\lambda t}U_W(t)f\ \right] d\mu \\
&\leq -e^{-\lambda t}\left\|WU_W(t)f\right\|
\end{aligned}$$

or

$$e^{-\lambda t}\left\|WU_W(t)f\right\| \leq -\frac{d}{dt}\left\|e^{-\lambda t}U_W(t)f\right\|. \tag{32}$$

It follows for $\lambda > 0$ that and

$$\int_0^{+\infty} e^{-\lambda t}\left\|WU_W(t)f\right\| dt \leq -\int_0^{+\infty}\frac{d}{dt}\left\|e^{-\lambda t}U_W(t)f\right\| dt = \|f\|.$$

Thus

$$\int_0^{+\infty} e^{-\lambda t}\left\|V_nU_{V_m}(t)f\right\| dt \leq \|f\|,\ \forall m \geqslant n$$

since $U_{V_m}(t) \leq U_{V_n}(t)$. Letting $m \to +\infty$, by monotone (decreasing) convergence we get

$$\int_0^{+\infty} e^{-\lambda t}\left\|V_nU_V(t)f\right\| dt \leq \|f\|$$

and then, by monotone (increasing) convergence, we obtain

$$\int_0^{+\infty} e^{-\lambda t}\left\|VU_V(t)f\right\| dt \leq \|f\|$$

which is nothing but

$$\left\|V(\lambda - T_V)^{-1}f\right\| \leq \|f\|$$

for $f \in D(T) \cap L^1_+(\Omega;\mu)$. Finally the density of $D(T) \cap L^1_+(\Omega;\mu)$ in $L^1_+(\Omega;\mu)$ and the fact that $L^1(\Omega;\mu) = L^1_+(\Omega;\mu) - L^1_+(\Omega;\mu)$ show that $V(\lambda - T_V)^{-1}$ is a bounded operator or equivalently V is T_V-bounded. ■

We show now, under a suitable assumption, that the above smoothing effect of $(\lambda - T_V)^{-1}$ is inherited by $(U_V(t))_{t\geqslant 0}$.

Theorem 2 *Let V satisfy (6)(8). If (12) is satisfied then*

$$U_V(t)f \in D(V) \text{ and } \|VU_V(t)f\| \leq \; c_t \|f\| \quad \forall f \in L^1(\Omega;\mu) \tag{33}$$

where

$$c_t := \liminf_{\varepsilon \to 0} \left\| \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega;\mu)}.$$

Proof. Let $f \in L^1_+(\Omega;\mu)$. We start from (32) with $\lambda = 0$

$$\|WU_W(t)f\| \leq -\frac{d}{dt}\|U_W(t)f\|.$$

Then

$$\int_a^b \|V_n U_{V_n}(s)f\| \, ds \leq \|U_{V_n}(a)f\| - \|U_{V_n}(b)f\|.$$

In particular

$$\int_a^b \|V_n U_{V_m}(s)f\| \, ds \leq \|U_{V_n}(a)f\| - \|U_{V_n}(b)f\| \quad \forall m \geqslant n$$

so that (by the construction of $(U_V(t))_{t\geqslant 0}$) letting $m \to +\infty$

$$\int_a^b \|V_n U_V(s)f\| \, ds \leq \|U_{V_n}(a)f\| - \|U_{V_n}(b)f\| \quad \forall n$$

and letting $n \to +\infty$ (by monotone convergence theorem)

$$\begin{aligned}
\int_a^b \|VU_V(s)f\| \, ds &\leq \|U_V(a)f\| - \|U_V(b)f\| \\
&= \int U_V(a)f - \int U_V(b)f \\
&= \int \left(U_V^*(a)1 - U_V^*(b)1\right) f.
\end{aligned}$$

In particular

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \|VU_V(s)f\| \, ds \leq \int \frac{U_V^*(t)1 - U_V^*(t+\varepsilon)1}{\varepsilon} f \quad (\varepsilon > 0)$$

i.e.

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\{V(x)>0\}} V(x)\left(U_V(s)f\right)(x)\mu(dx)ds \leq \int \frac{U_V^*(t)1 - U_V^*(t+\varepsilon)1}{\varepsilon} f.$$

We choose an *arbitrary* $\delta > 0$. Then (using (6))

$$\begin{aligned} & \int_{\{V(x)>0\}} V(x) \left(U_V(s)f\right)(x)\mu(dx) \\ = & \sum_{k\in\mathbb{Z}} \int_{\left\{(1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}} V(x) \left(U_V(s)f\right)(x)\mu(dx) \\ \geqslant & \sum_{k\in\mathbb{Z}} (1+\delta)^k \int_{\left\{(1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}} \left(U_V(s)f\right)(x)\mu(dx). \end{aligned}$$

It follows, for arbitrary $M > 0$, that

$$\begin{aligned} & \sum_{|k|\le M} (1+\delta)^k \frac{1}{\varepsilon} \int_t^{t+\varepsilon} ds \int_{\left\{(1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}} \left(U_V(s)f\right)(x)\mu(dx) \\ \le & \int \frac{U_V^*(t)1 - U_V^*(t+\varepsilon)1}{\varepsilon} f \le \left\| \frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \right\|_{L^\infty(\Omega;\mu)} \|f\| \quad (34) \end{aligned}$$

so that, knowing that

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} U_V(s) f ds \to U_V(t) f \quad (\varepsilon \to 0_+) \text{ in } L^1(\Omega;\mu),$$

and passing to the limit in (34) as $\varepsilon \to 0_+$ we get

$$\sum_{|k|\le M} (1+\delta)^k \int_{\left\{(1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}} \left(U_V(t)f\right)(x)\mu(dx) \le c_t \|f\| \qquad \forall M > 0$$

or equivalently

$$\sum_{k\in\mathbb{Z}} \int_{\left\{(1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}} (1+\delta)^k \left(U_V(t)f\right)(x)\mu(dx) \le c_t \|f\|.$$

On the other hand, on the set

$$\left\{x; (1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}$$

we have

$$\frac{V(x)}{1+\delta} < (1+\delta)^k$$

so

$$\frac{1}{1+\delta} \sum_{k\in\mathbb{Z}} \int_{\left\{(1+\delta)^k \le V(x) < (1+\delta)^{k+1}\right\}} V(x) \left(U_V(t)f\right)(x)\mu(dx) \le c_t \|f\|$$

i.e.
$$\frac{1}{1+\delta}\int_{\{V(x)>0\}} V(x)\left(U_V(t)f\right)(x)\mu(dx) \leq c_t \|f\|$$
or
$$\frac{1}{1+\delta}\|VU_V(t)f\| \leq c_t \|f\|.$$
It follows that
$$\|VU_V(t)f\| \leq c_t \|f\|$$
since $\delta > 0$ is arbitrary. For arbitrary $f \in L^1(\Omega;\mu)$, the positivity of V and $U_V(t)$ implies
$$\|VU_V(t)f\| \leq \|VU_V(t)\,|f|\| \leq \ c_t \,\||f|\| = c_t \|f\|$$
and ends the proof. ■

We deduce immediately:

Corollary 3 *Let V satisfy (6)(8). If (12) is satisfied then*
$$\int_{\{V>M\}}(U_V(t)f)\mu(dx) \leq \frac{c_t\|f\|}{M}, \forall\ f \in L^1_+(\Omega;\mu),\ \forall\ M>0,\ \forall t>0. \quad (35)$$

Remark 4 *The weak type estimate (35) was obtained previously in a direct way in [49] for almost all $t > 0$ under the additional assumption that $L^1(\Omega;\mathcal{A},\mu)$ is separable.*

It is worth to analyze the key Assumption (12).

Proposition 5 *If (15) is satisfied then so is (12). If (16) is satisfied (in particular if $1 \in \overline{D\left((T_V)^*\right)}$) then (15) and (12) are equivalent.*

Proof. Note that (15) amounts to
$$\forall t>0,\ \lim_{\varepsilon\to 0}\frac{U_V^*(t+\varepsilon)1 - U_V^*(t)1}{\varepsilon} \text{ exists}$$
in the *weak star topology* of $L^\infty(\Omega;\mu)$ which in turn implies the boundedness of $\varepsilon^{-1}\left\|U_V^*(t+\varepsilon)1 - U_V^*(t)1\right\|_{L^\infty(\Omega;\mu)}$ for $\varepsilon \in]0,1]$ by the uniform boundedness principle and implies (12). Conversely, let (12) be satisfied, i.e.
$$\lim\inf_{\varepsilon\to 0_+}\left\|\frac{U_V^*(\varepsilon)g_t - g_t}{\varepsilon}\right\|_{L^\infty(\Omega;\mu)} < +\infty \quad (36)$$

where $g_t := U_V^*(t)1$ $(t > 0)$. The subspace of $L^\infty(\Omega;\mu)$ of *strong continuity* of $(U_V^*(t))_{t\geqslant 0}$ is nothing but $\overline{D((T_V)^*)}$ (and is invariant under $(U_V^*(t))_{t\geqslant 0}$) so that (16) is equivalent to

$$g_t \in \overline{D((T_V)^*)}. \tag{37}$$

Finally (37) and ([9] Theorem 2.1.4 (c) p. 91) imply that $g \in \overline{D((T_V)^*)}$. ■

We do not consider here the question whether $U^*(t)1 \in \overline{D(T^*)}$ $(\forall t > 0)$ can imply $U_V^*(t)1 \in \overline{D((T_V)^*)}$ $(\forall t > 0)$? Note that if $(U_V(t))_{t\geqslant 0}$ is operator norm continuous then so is $(U_V^*(t))_{t\geqslant 0}$ and of course (16) is satisfied or equivalently $U_V^*(t)1 \in \overline{D((T_V)^*)}$ $(t > 0)$. We note also that if Ω is a locally compact space endowed with a Radon measure μ and if $(U_V^*(t))_{t\geqslant 0}$ leaves invariant (and is strongly continuous on) the subspace of bounded and uniformly continuous functions then of course $1 \in \overline{D((T_V)^*)}$ and consequently $U_V^*(t)1 \in \overline{D((T_V)^*)}$ $(\forall t > 0)$. (See [54] for the Feller properties of $(U_V^*(t))_{t\geqslant 0}$.)

Note that (12) is also satisfied if

$$(0,+\infty) \ni t \to U_V(t) \in \mathcal{L}(L^1(\Omega;\mu)) \text{ is locally lipschitz} \tag{38}$$

since

$$\begin{aligned} \left\| U_V^*(t+\varepsilon)1 - U_V^*(t)1 \right\|_{L^\infty(\Omega;\mu)} &\leq \left\| U_V^*(t+\varepsilon) - U_V^*(t) \right\|_{\mathcal{L}(L^\infty(\Omega;\mu))} \\ &= \left\| U_V(t+\varepsilon) - U_V(t) \right\|_{\mathcal{L}(L^1(\Omega;\mu))}. \end{aligned}$$

Note finally that the condition (38) is weaker than a differentiability condition on the perturbed C_0-semigroup $(U_V(t))_{t\geqslant 0}$ because the differentiability of a bounded C_0-semigroup $(S(t))_{t\geq 0}$ in a Banach space X is equivalent to *global* Lipschitz conditions

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0;\ \left\| S(t) - S(s) \right\|_{\mathcal{L}(X)} \leq C_\varepsilon \left| t - s \right|,\ \forall t, s \geqslant \varepsilon,$$

(see e.g. [29] Lemma 2.1).

Weighted shift semigroups on $L^1(\mathbb{R}, dx)$ give us some insight into the nature of (12).

Proposition 6 *Let $V \in L^1_{loc}(\mathbb{R})$ and let $(U(t))_{t\geqslant 0}$ be the translation C_0-semigroup on $L^1(\mathbb{R}, dx)$*

$$f \to U(t)f = f(x-t), \quad f \in L^1(\mathbb{R}, dx). \tag{39}$$

The perturbed C_0-semigroup $(U_V(t))_{t\geqslant 0}$

$$U_V(t)f = e^{-\int_{x-t}^{x} V(s)ds} f(x-t)$$

satisfies (12) if and only if

$$\liminf_{\varepsilon\to 0_+} \left\| \frac{\zeta_{t+\varepsilon}-\zeta_t}{\varepsilon} \right\|_{L^\infty(\mathbb{R})} < +\infty,\ (t>0) \tag{40}$$

where $\zeta_t \in L^\infty(\mathbb{R})$ is the function

$$\zeta_t : \mathbb{R} \ni y \to e^{-\int_y^{y+t} V(s)ds}.$$

In particular (12) is satisfied if

$$(0,+\infty) \ni t \to \zeta_t \in L^\infty(\mathbb{R}, dx) \text{ is locally lipschitz.} \tag{41}$$

Condition (16) amounts to

$$(0,+\infty) \ni t \to \zeta_t \in L^\infty(\mathbb{R}, dx) \text{ is continuous.}$$

Proof. A change of variable shows that

$$\int_{\mathbb{R}} U_V(t)f = \int_{\mathbb{R}} e^{-\int_y^{y+t} V(s)ds} f(y)dy$$

so that

$$U_V^*(t)1 = e^{-\int_y^{y+t} V(s)ds} = \zeta_t(y).$$

In particular

$$\|U_V^*(t+\varepsilon)1 - U_V^*(t)1\|_{L^\infty(\mathbb{R})} = \left\|\zeta_{t+\varepsilon} - \zeta_t\right\|_{L^\infty(\mathbb{R})}$$

which ends the proof. ■

Thus, under e.g. (41), $(U_V(t))_{t\geqslant 0}$ satisfies (12) although it is *neither* differentiable *nor* operator norm-continuous. Actually, we can check directly (20) for the translation C_0-semigroup above without appealing to (12).

Proposition 7 *Let $V \in L^1_{loc}(\mathbb{R})$. Let $(U(t))_{t\geqslant 0}$ be the translation C_0-semigroup (39).*

(i) The smoothing effect (20) holds if and only if

$$y \to V(t+y)e^{-\int_y^{y+t} V(s)ds} \text{ is bounded } (t>0). \tag{42}$$

(ii) If V is differentiable, bounded away from zero and if $\frac{V'}{V}$ is bounded then (42) is satisfied.

Proof. (i) follows from a simple change of variable since

$$\int_{\mathbb{R}} V(x)\left(U_V(t)f\right)(x)dx = \int_{\mathbb{R}} V(t+y)e^{-\int_y^{y+t} V(s)ds} f(y)dy.$$

(ii) We note that for all $u, v \in \mathbb{R}$

$$\frac{V(u)}{V(v)} = e^{\int_v^u \frac{V'(s)}{V(s)}ds}$$

so

$$\frac{V(u)}{V(v)} \leq e^{C|u-v|}$$

where $C = \sup \frac{V'}{V}$. On the other hand, there exists $x \in [y, y+t]$ (depending on y and t) such that

$$\frac{1}{t}\int_y^{y+t} V(s)ds = V(x)$$

whence (using $\alpha := \sup_{z\geq 0} ze^{-z}$)

$$\begin{aligned} V(t+y)e^{-\int_y^{y+t} V(s)ds} &= \frac{V(t+y)}{V(x)} V(x)e^{-tV(x)} \\ &\leq \alpha e^{Ct} \quad \forall y \in \mathbb{R} \end{aligned}$$

since $|x-(t+y)| \leq t$. ■

Remark 8 *In Proposition 7 (ii), V may have e.g. a polynomial growth at infinity.*

The operator norm continuity of $(U_V(t))_{t\geqslant 0}$ is of course a natural mean to translate compactness properties from the resolvent $(\lambda - T_V)^{-1}$ to the semigroup $U_V(t)$ (see [57] Theorem 3.3, p. 48). However, it is an open problem to decide whether the operator norm continuity of a substochastic C_0-semigroup $(U(t))_{t\geqslant 0}$ is inherited by $(U_V(t))_{t\geqslant 0}$ regardless of V. This problem is not covered by the paper [38] which deals with a *special* class of unbounded perturbations preserving immediate norm continuity of C_0-semigroups. We provide here a solution to this open problem.

Theorem 9 *Let V satisfy (6)(8). Let $V_n := V \wedge n$.*

(i) Then for all finite $C > 0$ and all $f \in L_+^1(\Omega;\mu)$

$$\sup_{t\leq C}\left\|e^{t(T-V_n)}f - U_V(t)f\right\| \leq e^C \left\|\left[V-V_n\right](1-T_V)^{-1}f\right\|. \tag{43}$$

In particular, if $(U(t))_{t\geqslant 0}$ is operator norm continuous and if

$$\left\| [V - V_n]\,(1-T_V)^{-1} \right\|_{\mathcal{L}(L^1(\Omega;\mu))} \to 0 \text{ as } n \to +\infty \tag{44}$$

then $(U_V(t))_{t\geqslant 0}$ is also operator norm continuous.

(ii) In particular, let $(1-T_V)^{-1}$ be an integral operator with kernel $G_V(x,y)$. If $(U(t))_{t\geqslant 0}$ is operator norm continuous and if

$$\sup_{y\in\Omega} \int_{\{V\geqslant n\}} G_V(x,y)V(x)\mu(dx) \to 0 \text{ as } n \to +\infty \tag{45}$$

then $(U_V(t))_{t\geqslant 0}$ is also operator norm continuous.

Proof. Note first that both V and V_n are T_V-bounded so that the sequence $\left\{ [V - V_n]\,(1-T_V)^{-1} \right\}_n$ of bounded operators converges strongly to zero. According to the general theory $e^{t(T-V_n)}f \to U_V(t)f$ for all $f \in L^1(\Omega;\mu)$ uniformly in $t \in [0, C]$. We start with the Duhamel formula (for a positive bounded perturbation) and $f \in L^1_+(\Omega;\mu)$

$$e^{t(T-V_n)}f = e^{t(T-V_{n+k})}f + \int_0^t e^{(t-s)(T-V_{n+k})}\,[V_{n+k} - V_n]\, e^{s(T-V_{n+k})} f ds.$$

By letting $k \to +\infty$, $V_{n+k}(x) - V_n(x) \to V(x) - V_n(x)$ a.e. and then

$$e^{t(T-V_n)}f = U_V(t)f + \int_0^t U_V(t-s)\,[V - V_n]\, U_V(s) f ds.$$

The *additivity of the norm* on the positive cone shows that

$$\begin{aligned}
\left\| e^{t(T-V_n)}f - U_V(t)f \right\| &= \left\| \int_0^t U_V(t-s)\,[V - V_n]\, U_V(s) f ds \right\| \\
&= \int_0^t \left\| U_V(t-s)\,[V - V_n]\, U_V(s) f \right\| ds \\
&\leq \int_0^t \left\| [V - V_n]\, U_V(s) f \right\| ds = \left\| \int_0^t [V - V_n]\, U_V(s) f ds \right\| \\
&= \left\| [V - V_n] \int_0^t U_V(s) f ds \right\| \leq \left\| [V - V_n] \int_0^C U_V(s) f ds \right\| \\
&\leq e^C \left\| [V - V_n] \int_0^C e^{-s} U_V(s) f ds \right\|
\end{aligned}$$

for all $t \leq C$ where $C > 0$ is arbitrary. Hence

$$\sup_{t\leq C} \left\| e^{t(T-V_n)}f - U_V(t)f \right\| \leq e^C \left\| [V - V_n]\,(1-T_V)^{-1} f \right\|,\ \forall f \in L^1_+(\Omega;\mu)$$

and

$$\sup_{t\leq C}\left\|e^{t(T-V_n)}-U_V(t)\right\|\leq e^C\left\|[V-V_n](1-T_V)^{-1}\right\|.$$

Finally, if $(U(t))_{t\geqslant 0}$ is operator norm continuous then so is $\left(e^{t(T-V_n)}\right)_{t\geq 0}$ because V_n is a bounded perturbation [60] so that the last operator norm estimate ends the proof of (i). If $(1-T_V)^{-1}$ is an integral operator with kernel $G_V(x,y)$ then an elementary calculation shows that

$$\left\|[V-V_n](1-T_V)^{-1}\right\|_{\mathcal{L}(L^1(\Omega))}=\sup_{y\in\Omega}\int_{\{V\geqslant n\}}G_V(x,y)V(x)\mu(dx)$$

and this combined with (i) end the proof of (ii). ■

Remark 10 *Condition (45) is of course satisfied if*

$$\sup_{y\in\Omega}\int_{\{V\geqslant n\}}G(x,y)V(x)\mu(dx)\to 0 \text{ as } n\to+\infty \tag{46}$$

where $G(x,y)$ is the kernel of $(1-T)^{-1}$. In particular, if $(1-T)^{-1}\in\mathcal{L}(L^1(\Omega),L^p(\Omega))$ for some $p>1$ and if $V\in L^{p^*}(\Omega)$ (p^* is the conjugate exponent of p) then (46) is satisfied.

We give now a stability property for a suitable class of differentiable C_0-semigroups $(U(t))_{t\geqslant 0}$ and suitable perturbations V.

Theorem 11 *Let $(U(t))_{t\geqslant 0}$ be a class-$\mathcal{P}$ differentiable C_0-semigroup and let V belong to its generalized Kato class potentials in the sense (19). Then $(U_V(t))_{t\geqslant 0}$ is class-$\mathcal{P}$ differentiable.*

Proof. Let $V_n:=V\wedge n$ and let $\omega>0$ be such that

$$r_\sigma\left[V(\omega-T)^{-1}\right]<1.$$

Since $(U(t))_{t\geqslant 0}$ is positive then it is easy to see that for any integer k

$$\left\|\left(V_n(\omega+is-T)^{-1}\right)^k\right\|\leq\left\|\left(V_n(\omega-T)^{-1}\right)^k\right\|\leq\left\|\left(V(\omega-T)^{-1}\right)^k\right\|$$

so that

$$r_\sigma\left[V_n(\omega+is-T)^{-1}\right]\leq r_\sigma\left[V(\omega-T)^{-1}\right]<1\ \ \forall s\in\mathbb{R},\ \forall n.$$

Thus

$$(\omega + is - (T - V_n))^{-1} f = (\omega + is - T)^{-1} \sum_{k=0}^{\infty} (-1)^k \left(V_n(\omega + is - T)^{-1}\right)^k f$$

and

$$\begin{aligned}
\left\| (\omega + is - (T - V_n))^{-1} f \right\| &\leq \left\| (\omega + is - T)^{-1} \right\| \sum_{k=0}^{\infty} \left\| \left(V_n(\omega + is - T)^{-1}\right)^k \right\| \|f\| \\
&\leq \left\| (\omega + is - T)^{-1} \right\| \sum_{k=0}^{\infty} \left\| \left(V(\omega - T)^{-1}\right)^k \right\| \|f\| .
\end{aligned}$$

On the other hand, by construction (see (7)), $e^{t(T-V_n)} \to e^{tT_V}$ strongly as $n \to \infty$ so that

$$(\omega + is - (T - V_n))^{-1} f = \int_0^{\infty} e^{-(\omega+is)t} e^{t(T-V_n)} f dt$$

implies

$$(\omega + is - (T - V_n))^{-1} f \to (\omega + is - T_V)^{-1} f \quad \text{as } n \to \infty$$

and

$$\left\| (\omega + is - T_V)^{-1} \right\| \leq \left\| (\omega + is - T)^{-1} \right\| \sum_{k=0}^{\infty} \left\| \left(V(\omega - T)^{-1}\right)^k \right\| .$$

Finally $\lim_{|s| \to \infty} \ln |s| \left\| (\omega + is - T_V)^{-1} \right\|$ is than or equal to

$$\left(\sum_{k=0}^{\infty} \left\| \left(V(\omega - T)^{-1}\right)^k \right\| \right) \lim_{|s| \to \infty} \ln |s| \left\| (\omega + is - T)^{-1} \right\| = 0$$

which ends the proof. ■

In the theorem above, it is not clear whether we can remove the assumption that V belongs to the generalized Kato class of $(U(t))_{t \geqslant 0}$. We end this section with a helpful tool.

Lemma 12 *Let $(V(t))_{t \geq 0}$ be a C_0-semigroup on $L^1(\Omega; \mu)$ with generator G. If the resolvent $(\lambda - G)^{-1}$ is a weakly compact operator for some (or equivalently all) $\lambda \in \rho(G)$ then $(\lambda - G)^{-1}$ is a compact operator for all $\lambda \in \rho(G)$.*

Proof. The resolvent identity

$$(\lambda - G)^{-1} - (\mu - G)^{-1} = (\mu - \lambda)(\lambda - G)^{-1}(\mu - G)^{-1}; \quad \lambda, \mu \in \rho(G)$$

shows that the weak compactness of $(\lambda - G)^{-1}$ implies the the weak compactness of $(\mu - G)^{-1}$. By the classical Dunford-Pettis' theorem (see e.g. [1] Corollary 5.88, p. 344) the product of two weakly compact operators on $L^1(\Omega;\mu)$ is a compact operator so that

$$\left\|(\lambda - G)^{-1} - (\mu - \lambda)(\lambda - G)^{-1}(\mu - G)^{-1}\right\| = \left\|(\mu - G)^{-1}\right\| \to 0 \text{ as } \mu \to +\infty$$

shows that $(\lambda - G)^{-1}$ is a compact operator. ∎

3 Compactness results on abstract $L^1(\Omega; \mathcal{A}, \mu)$ spaces

As pointed out in Section 1, for any linear operator $O \in \mathcal{L}(L^1(\Omega;\mu))$ and for any measurable subset $\Xi \subset \Omega$, the notation

$$O : L^1(\Omega;\mu) \to L^1(\Xi;\mu)$$

refers to the operator

$$L^1(\Omega;\mu) \ni f \to [Of]_{|\Xi} \in L^1(\Xi;\mu)$$

where $[Of]_{|\Xi}$ is the *restriction* of Of to the subset Ξ. We start with:

Theorem 13 *Let V satisfy (6)(8). Let $(U(t))_{t\geqslant 0}$ be a sub-stochastic C_0-semigroup on $L^1(\Omega;\mu)$ with generator T. Then T_V is resolvent compact if and only if for all $M > 0$*

$$(\lambda - T_V)^{-1} : L^1(\Omega;\mu) \to L^1(\Omega_M;\mu) \text{ is weakly compact.} \tag{47}$$

A sufficient condition for (47) to hold is that

$$(\lambda - T)^{-1} : L^1(\Omega;\mu) \to L^1(\Omega_M;\mu) \text{ is weakly compact.} \tag{48}$$

Proof. According to Lemma 12, it suffices to show that T_V is resolvent weakly compact. Let $f = (\lambda - T_V)^{-1} g$ with $\lambda > s(T_V)$ ($g \in B$) where B is the unit ball of $L^1(\Omega;\mu)$. Since $D(T_V) \subset D(V)$ and V is T_V-bounded (Lemma 1) then there exists a constant $c > 0$ such that $\|Vf\| \leq c\,\|g\|$ so that

$$\begin{aligned} M\int_{\{V(x)\geqslant M\}} |f(x)|\,\mu(dx) &\leq \int_{\{V(x)\geqslant M\}} V(x)\,|f(x)|\,\mu(dx) \\ &\leq \int V(x)\,|f(x)|\,\mu(dx) \leq c, \ \forall g \in B \end{aligned}$$

so that $\int_{\{V(x)\geqslant M\}} |f(x)|\,\mu(dx) \to 0$ as $M \to +\infty$ uniformly in $g \in B$. Thus we have decomposed $f = (\lambda - T_V)^{-1}g$ as $f1_{\Omega_M} + f1_{\Omega_M^c}$ where $f1_{\Omega_M^c}$ can be made as small in L^1-norm as we want (uniformly in $g \in B$) and $f1_{\Omega_M}$ is a relatively weakly compact set by (47). This shows the first claim. Finally, the domination

$$(\lambda - T_V)^{-1}_{|\Omega_M} \leq (\lambda - T)^{-1}_{|\Omega_M}$$

shows that (48) implies (47). ■

Remark 14 *If the sublevel sets Ω_M have finite μ-measure then Condition (48) is automatically satisfied provided that*

$$(\lambda - T)^{-1} \in \mathcal{L}(L^1(\Omega;\mu),\ L^p(\Omega;\mu))$$

for some $p > 1$. This follows from the fact that the embedding of $L^p(\Omega_M;\mu)$ into $L^1(\Omega_M;\mu)$ is weakly compact (i.e. a bounded subset of $L^p(\Omega_M;\mu)$ is an equi-integrable subset of $L^1(\Omega_M;\mu)$).

We complement Theorem 13 with:

Theorem 15 *Let V satisfy (6)(8). Let $(U(t))_{t\geqslant 0}$ be a sub-stochastic C_0-semigroup on $L^1(\Omega;\mu)$ with generator T. We assume that for $M > 0$ and $t > 0$*

$$U(t) : L^1(\Omega;\mu) \to L^1(\Omega_M;\mu) \text{ is weakly compact } (t > 0) \tag{49}$$

Then:

(i) T_V is resolvent compact.

(ii) If (12) is satisfied then $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup.

Proof. Let $P_{\Omega_M} : L^1(\Omega;\mu) \to L^1(\Omega_M;\mu)$ be the restriction operator. Note that

$$P_{\Omega_M}(\lambda - T)^{-1} = P_{\Omega_M}\int_0^{+\infty} e^{-\lambda t}U(t)dt = \lim_{\varepsilon\to 0} P_{\Omega_M}\int_\varepsilon^{\varepsilon^{-1}} e^{-\lambda t}U(t)dt$$

where the convergence holds in operator norm. Let us show that $P_{\Omega_M}(\lambda - T)^{-1}$ is weakly compact. It suffices to show that

$$P_{\Omega_M}\int_\varepsilon^{\varepsilon^{-1}} e^{-\lambda t}U(t)dt = \int_\varepsilon^{\varepsilon^{-1}} e^{-\lambda t}P_{\Omega_M}U(t)dt$$

is a weakly compact operator. This is a strong integral (not a Bochner integral) of a bounded, strongly continuous $W(L^1(\Omega;\mu), L^1(\Omega_M;\mu))$-valued mapping where $W(L^1(\Omega;\mu), L^1(\Omega_M;\mu))$ is the Banach space of weakly compact operators from $L^1(\Omega;\mu)$ into $L^1(\Omega_M;\mu)$. By [66] or [43]

$$\int_{\varepsilon}^{\varepsilon^{-1}} e^{-\lambda t} P_{\Omega_M} U(t) dt$$

is a weakly compact operator. Then the first claim is a consequence of Theorem 13.

(ii) We choose an arbitray $\bar{t} > 0$. Let $f = U_V(\bar{t})g$ with $g \in B$ the unit ball of $L^1(\Omega;\mu)$. By corollary 3 $\int_{\{V(x)\geqslant M\}} |f(x)| \, \mu(dx) \to 0$ as $M \to +\infty$ uniformly in $g \in B$. On the other hand

$$|f| = \left|U_V(\bar{t})g\right| \leq U_V(\bar{t}) \, |g| \leq U(\bar{t}) \, |g|$$

so that, by (49), the restriction to Ω_M of $\left\{U_V(\bar{t})g;\ g \in B\right\}$ is relatively weakly compact by domination and then, by arguing as in the proof of Theorem 13, one sees that $\left\{U_V(\bar{t})g;\ g \in B\right\}$ is a relatively weakly compact subset of $L^1(\Omega;\mu)$, i.e. $U_V(t)$ is a weakly compact operator for all $t \geqslant \bar{t}$ and consequently for all $t > 0$. Actually, $U_V(t)$ is a compact operator for all $t > 0$ since $U_V(t) = U_V(\frac{t}{2})U_V(\frac{t}{2})$ and the product of two weakly compact operators on $L^1(\Omega;\mu)$ is a compact operator (see e.g. [1] Corollary 5.88, p. 344). ■

Remark 16 *If the sublevel sets Ω_M have finite μ-measure then Condition (49) is automatically satisfied provided that*

$$U(t) \in L(L^1(\Omega;\mu),\ L^p(\Omega;\mu)) \quad (t > 0)$$

for some $p > 1$. This is the case e.g. for ultracontractive (in the sense (5)) symmetric Markov C_0-semigroups $(U(t))_{t\geqslant 0}$ since

$$U(t) \in \mathcal{L}(L^1(\Omega;\mu), L^2(\Omega;\mu)) \quad (t > 0).$$

Since the compactness of $(\lambda - T_V)^{-1}$ is equivalent to the compactness of $U_V(t)$ for $t > 0$ if $(U_V(t))_{t\geqslant 0}$ is operator norm continuous (see e.g. [57] Theorem 3.3, p. 48) then we have:

Corollary 17 *Let V satisfy (6)(8). Let $(U_V(t))_{t\geqslant 0}$ be operator norm continuous. If (48) is satisfied then $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup.*

It is not difficult to see that $(\lambda - T_V)^{-1}$ is compact if and only if $\int_0^t U_V(s)ds$ is for all $t \geqslant 0$ (the argument holds for general C_0-semigroups in Banach spaces). Thus Theorem 13 implies that

$$\int_0^t U_V(s)ds \text{ is a compact operator on } L^1(\Omega;\mu)$$

under Assumption (48) *only.*

If $(U(t))_{t\geqslant 0}$ is a *sub-Markov* C_0-semigroup (i.e. acts in all L^p spaces as a positive contraction semigroup), we denote it by $(U^p(t))_{t\geq 0}$ as a C_0-semigroup acting on $L^p(\Omega;\mu)$ and denote by T^p its generator. As in the L^1 case, we define the perturbed C_0-semigroup $\left(U_V^p(t)\right)_{t\geq 0}$ with generator T_V^p and $\left(U_V^p(t)\right)_{t\geq 0}$ is strongly continuous if and only if $(U_V(t))_{t\geqslant 0}$ is ([71] Proposition 3.1). Then using the compactness interpolation theorem for σ-finite measures (see e.g. [11] Theorem 1.6.1, p. 35) we obtain immediately:

Corollary 18 *Let V satisfy (6)(8). Let $(U(t))_{t\geqslant 0}$ be a sub-Markov C_0-semigroup and let μ be σ-finite. Then:*

(i) If (48) is satisfied then T_V^p is resolvent compact in $L^p(\Omega;\mu)$. If additionnaly $(U_V(t))_{t\geqslant 0}$ is operator norm continuous (on $L^1(\Omega;\mu)$) then the C_0-semigroups $\left\{U_V^p(t); t\geqslant 0\right\}$ are compact in $L^p(\Omega;\mu)$.

(ii) If (12)(49) are satisfied then the C_0-semigroups $\left(U_V^p(t)\right)_{t\geq 0}$ are compact in $L^p(\Omega;\mu)$.

A more precise result can be derived in the self-adjoint case:

Corollary 19 *Let V satisfy (6)(8). Let $(U(t))_{t\geqslant 0}$ be a symmetric sub-Markov C_0-semigroup and let μ be σ-finite. If (48) is satisfied then the C_0-semigroup $\left(U_V^p(t)\right)_{t\geq 0}$ is compact in $L^p(\Omega;\mu)$ for $p>1$.*

Proof. By Corollary 18, T_V^2 is resolvent compact. It follows that the self-adjoint C_0-semigroup $\left(U_V^2(t)\right)_{t\geq 0}$ itself is also compact for $t>0$. Then, an interpolation argument shows that $\left(U_V^p(t)\right)_{t\geq 0}$ is compact $(t>0)$ for all $p>1$. ■

Remark 20 *Note that under the assumptions of Corollary 19, the C_0-semigroup $(U_V(t))_{t\geqslant 0}$ need not be compact on $L^1(\Omega;\mu)$.*

We show now that the basic assumption (49) is stable by subordination. We recall first some notions on subordinate C_0-semigroups. Let $f \in C^\infty((0,+\infty))$ be a Bernstein function, i.e.

$$f \geqslant 0,\ (-1)^k \frac{d^k f(x)}{dx^k} \leq 0 \ \ \forall k \in \mathbb{N}.$$

It is characterized by the representation

$$e^{-tf(x)} = \int_0^{+\infty} e^{-xs}\eta_t(ds) \ (t > 0)$$

where $(\eta_t)_{t\geqslant 0}$ is a convolution C_0-semigroup of sub-probabiity measures on $[0, +\infty)$ (see e.g. [31] Theorem 3.9.7, p. 177). Let $(U(t))_{t\geqslant 0}$ be a contraction C_0-semigroup. We can define (see [31] Chapter 4 for the details) the so-called subordinate C_0-semigroup $\left(U^f(t)\right)_{t\geqslant 0}$ (in the sense of Bochner) acting as

$$\varphi \in L^1(\mathbb{R}^N) \to U^f(t)\varphi = \int_0^{+\infty} (U(s)\varphi)\eta_t(ds) \in L^1(\mathbb{R}^N).$$

Theorem 21 *Let $(U(t))_{t\geqslant 0}$ be a positive contraction C_0-semigroup on $L^1(\Omega; \mu)$ satisfying (49). Let f be a Bernstein function such that*

$$\lim_{x\to+\infty} f(x) = +\infty. \tag{50}$$

Then the subordinate C_0-semigroup $\left(U^f(t)\right)_{t\geqslant 0}$ satisfies also (49).

Proof. Note first that (50) (i.e. $e^{-tf(x)} \to 0$ as $x \to +\infty \ \ (t > 0)$) amounts to $\eta_t(\{0\}) = 0 \ \forall t > 0$. This implies that

$$\left\| \int_\varepsilon^{\varepsilon^{-1}} U(s)\eta_t(ds) - U^f(t) \right\| \leq \eta_t([0, \varepsilon[) + \eta_t(]\varepsilon^{-1}, +\infty[) \to 0 \ \text{ as } \ \varepsilon \to 0,$$

so that

$$\left\| \int_\varepsilon^{\varepsilon^{-1}} P_{\Omega_M} U(s)\eta_t(ds) - P_{\Omega_M} U^f(t) \right\| \to 0 \ \text{ as } \ \varepsilon \to 0.$$

It suffices to show that $\int_\varepsilon^{\varepsilon^{-1}} P_{\Omega_M} U(s)\eta_t(ds)$ is weakly compact. By assumption, $\forall s > 0$, $P_{\Omega_M} U(s)$ is weakly compact. Moreover

$$s > 0 \to P_{\Omega_M} U(s) \in \mathcal{L}(L^1(\mathbb{R}^N), L^1(\Omega_M))$$

is strongly continuous and bounded. It follows from [66] or [43] that the strong integral $\int_\varepsilon^{\varepsilon^{-1}} P_{\Omega_M} U(s)\eta_t(ds)$ is also weakly compact. ■

It seems that Assumption (50) is purely technical; indeed, see Theorem 27 below on convolution C_0-semigroups on $\mathbb{R}^N$ where this assumption is no longer necessary.

Remark 22 *It is not clear that Assumption (12) is stable by subordination. However, for some Bernstein functions f, the subordinate C_0-semigroup $\left(U^f(t)\right)_{t\geqslant 0}$ is always holomorphic (and thus (12) is satisfied by $\left(U_V^f(t)\right)_{t\geqslant 0}$) regardless of $(U(t))_{t\geqslant 0}$; we note also that if $(U(t))_{t\geqslant 0}$ is holomorphic then so is $\left(U^f(t)\right)_{t\geqslant 0}$ for any Bernstein function f; see [21] and references therein.*

Remark 23 *Let $L^1(\Omega;\mathcal{A},\mu)$ be separable. If $O : L^1(\Omega;\mu) \to L^1(\Omega;\mu)$ is such that*

$$1_{\Omega_M} O : L^1(\Omega;\mu) \ni f \to 1_{\Omega_M} O f \in L^1(\Omega_M;\mu)$$

is weakly compact then $1_{\Omega_M} O$ is (uniquely represented by) an integral operator with a measurable kernel (see the remark in [16] p. 508) and this clearly implies that O is an integral operator with a measurable kernel since $V(x) < +\infty$ a.e. Thus Condition (48) (resp. Condition (49)) implies that $(\lambda - T)^{-1}$ (resp. $U(t)$) is an integral operator with a measurable kernel. For instance, this is the case of ultracontractive symmetric Markov C_0-semigroups (see also [68] Corollary A.1.2).

4 Applications to perturbed convolution semigroups

Before continuing the general theory, we devote a section to specific compactness results on convolution C_0-semigroups on euclidean spaces. Let $\Xi \subset \mathbb{R}^N$ be a Borel subset. We say that Ξ is "thin at infinity" if

$$|\Xi \cap B(z;1)| \to 0 \text{ as } |z| \to \infty \tag{51}$$

where $B(z;1)$ is the ball with radius 1 centered at $z \in \mathbb{R}^N$ and $||$ refers to Lebesgue measure. We start with a basic result.

Lemma 24 *Let $h \in L^1_+(\mathbb{R}^N)$ and let*

$$H : L^1(\mathbb{R}^N) \ni \varphi \to \int_{\mathbb{R}^N} h(x-y)\varphi(y)dy \in L^1(\mathbb{R}^N)$$

be a convolution operator. Let $\Xi \subset \mathbb{R}^N$ be a Borel set. Then

$$H : L^1(\mathbb{R}^N) \to L^1(\Xi)$$

is compact if and only if

$$\sup_{y\in\mathbb{R}^N} \int_{\Xi\cap\{|x|\geqslant c\}} h(x-y)dx \to 0 \text{ as } c \to +\infty. \tag{52}$$

Moreover (52) is satisfied if Ξ is "thin at infinity".

Proof. We note first that the continuity of $y \in \mathbb{R}^N \to h^y(.) \in L^1(\mathbb{R}^N)$ (where $h^y(.) : x \to h(x-y)$ is the translation of $h(.)$ by a vector y) shows that $H : L^1(\mathbb{R}^N) \to L^1(\Xi)$ is compact for any bounded Borel set Ξ. On the other hand, if $H : \varphi \in L^1(\mathbb{R}^N) \to L^1(\Xi)$ is compact then

$$\left\| \chi_{\Xi \cap \{|x|>c\}} H \right\|_{\mathcal{L}(L^1(\mathbb{R}^N), L^1(\Xi))} \to 0 \ \text{ as } \ c \to +\infty$$

(we still denote by $\chi_{\Xi \cap \{|x|>c\}}$ the multiplication operator by the indicator function $\chi_{\Xi \cap \{|x|>c\}}$) because $\left\| \chi_{\{|x|>c\}} f \right\|_{L^1(\Xi)} \to 0$ as $c \to +\infty$ uniformly in f in a compact set of $L^1(\Xi)$, i.e. (52) holds. Conversely, under (52), $H : \varphi \in L^1(\mathbb{R}^N) \to L^1(\Xi)$ is a limit in operator norm (as $c \to +\infty$) of $\chi_{\Xi \cap \{|x| \leq c\}} H$ which is compact since $\Xi \cap \{|x| \leq c\}$ is bounded.

Let us show now that (52) is satisfied if Ξ is "thin at infinity". To show (52) it suffices that

$$\lim_{|y| \to +\infty} \int_{\Xi} h(x-y) dx = 0. \tag{53}$$

Indeed, let $\varepsilon > 0$ be arbitrary and let $D > 0$ be such that

$$\int_{\Xi} h(x-y) dx \leq \varepsilon \ \text{ for all } \ |y| > D.$$

It suffices to show that for any $D > 0$

$$\sup_{|y| \leq D} \int_{\Xi \cap \{|x| \geqslant c\}} h(x-y) dx \to 0 \text{ as } c \to +\infty$$

i.e.

$$\sup_{|y| \leq D} \int_{\Xi \cap \{|x| \geqslant c\}} h^y(x) dx \to 0 \text{ as } c \to +\infty. \tag{54}$$

Since $y \in \mathbb{R}^N \to h^y(.) \in L^1(\mathbb{R}^N)$ is continuous then

$$\{h^y(.);\ |y| \leq D\} \ \text{ is compact subset of } L^1(\mathbb{R}^N)$$

and consequently $\{h^y(.);\ |y| \leq D\}$ is an equi-integrable subset of $L^1(\mathbb{R}^N)$ so that (54) is true. It suffices now to show that (53) is satisfied if Ξ is "thin at infinity". We observe first that (51) is actually equivalent to

$$\forall R \geqslant 1,\ |\Xi \cap B(y; R)| \to 0 \ \text{ as } \ |y| \to \infty \tag{55}$$

where $B(y; R)$ is the ball with radius R centered at $y \in \mathbb{R}^N$. It suffices to observe that $|\Xi \cap B(y; R)| \leq \sum_{i=1}^{J_R} |\Xi \cap B(y_i; 1)|$ where we have covered

$B(y;R)$ by a finite number J_R (depending on R only) of balls $B(y_i;1)$ with radius 1. We write

$$\begin{aligned}\int_{\Xi} h(x-y)dx &= \int_{\Xi-y} h(z)dz = \int_{(\Xi-y)\cap B(0,R)} h(z)dz + \int_{(\Xi-y)\cap B(0,R)^c} h(z)dz \\ &\leq \int_{(\Xi-y)\cap B(0,R)} h(z)dz + \int_{B(0,R)^c} h(z)dz\end{aligned}$$

where $B(0;R)^c$ is the exterior of the ball $B(0;R)$. The invariance of Lebesgue measure by translation yields

$$|(\Xi-y)\cap B(0;R)| = |\Xi\cap B(y;R)|\,. \tag{56}$$

Finally, for any $\varepsilon>0$ we choose R large enough so that $\int_{B(0;R)^c} h(z)dz<\varepsilon$ and then $\int_{(\Xi-y)\cap B(0;R)} h(z)dz \to 0$ as $|y|\to+\infty$ by (55) and (56). ■

We consider now the convolution C_0-semigroup $(U(t))_{t\geqslant 0}$ with generator T introduced in Section 1

$$U(t):\varphi\in L^1(\mathbb{R}^N)\to\int\varphi(x-y)m_t(dy)\in L^1(\mathbb{R}^N) \tag{57}$$

where $\{m_t\}_{t\geq 0}$ are Radon sub-probability measures on $\mathbb{R}^N$ such that $m_0=\delta_0$, $m_t * m_s = m_{t+s}$ and $m_t\to m_0$ vaguely as $t\to 0_+$. Such convolution C_0-semigroups cover many examples of practical interest such as Gaussian semigroups, α-stable semigroups, relativistic Schrödinger semigroups, relativistic α-stable semigroup etc. (see [31] Chapter 3). This C_0-semigroup acts in all $L^p(\mathbb{R}^N)$ $(1\leq p<+\infty)$; in such spaces, we denote it by $(U^p(t))_{t\geqslant 0}$ and denote its generator by T^p. We recall that

$$(\lambda-T)^{-1}\varphi=\int\varphi(x-y)m^{\lambda}(dy)$$

where

$$\widehat{m^{\lambda}}(\zeta)=\int_0^{+\infty}e^{-\lambda t}\widehat{m_t}(\zeta)dt=\frac{1}{\lambda+F(\zeta)}.$$

Two kinds of assumptions can be used. Either

$$\exists\ p_t\in L^1_+(\mathbb{R}^N) \text{ such that } m_t(dy)=p_t(y)dy\ \ (t>0) \tag{58}$$

or

$$\exists\ G_\lambda\in L^1_+(\mathbb{R}^N) \text{ such that } m^{\lambda}(dy)=G_\lambda(y)dy. \tag{59}$$

Note that (59) is much weaker than (58) . Note also that (58) is satisfied if $e^{-tF(\zeta)}\in L^1(\mathbb{R}^N)$ $(t>0)$. As a consequence of Lemma 24 we have:

Theorem 25 *Let $(U(t))_{t\geqslant 0}$ be the convolution C_0-semigroup (57) on $L^1(\mathbb{R}^N)$. Let the sublevel sets Ω_M be "thin at infinity". If (59) is satisfied then T_V is resolvent compact on $L^1(\mathbb{R}^N)$. If (58) and (12) are satisfied then $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup on $L^1(\mathbb{R}^N)$.*

Since $\left(U^2(t)\right)_{t\geq 0}$ is self-adjoint for real characteristic exponent then Corollary 19 implies:

Corollary 26 *We assume that the characteristic exponent is real. Let (59) be satisfied and Ω_M be "thin at infinity". Then $\left(U_V^p(t)\right)_{t\geq 0}$ are compact C_0-semigroups on $L^p(\mathbb{R}^N)$ for all $p > 1$.*

We give now a subordination result. For any Bernstein function f, we denote by $\left(U^f(t)\right)_{t\geq 0}$ the subordinated C_0-semigroup (in the sense of Bochner) defined in Section 3 which is also a convolution C_0-semigroup with characteristic exponent $F^f = f \circ F$. We denote by $\left(U_V^f(t)\right)_{t\geq 0}$ the corresponding perturbed C_0-semigroup, i.e.

$$U_V^f(t) := (U^f)_V(t).$$

Theorem 27 *Let $(U(t))_{t\geqslant 0}$ be the convolution C_0-semigroup (57) on $L^1(\mathbb{R}^N)$. Let f be a Bernstein function and let $\left(U^f(t)\right)_{t\geqslant 0}$ be the corresponding subordinate C_0-semigroup. We assume that m_t are functions ($t > 0$). If (12) is satisfied by $\left(U^f(t)\right)_{t\geqslant 0}$ and if the sublevel sets Ω_M are "thin at infinity" then $\left(U_V^f(t)\right)_{t\geq 0}$ is a compact C_0-semigroup on $L^1(\mathbb{R}^N)$.*

Proof. Let $\left\{m_t^f\right\}_{t\geq 0}$ be the Radon sub-probability measures corresponding to the convolution C_0-semigroup $\left(U^f(t)\right)_{t\geqslant 0}$. We have

$$U(t) = \varphi * m_t$$

and

$$U^f(t)\varphi = \int_0^{+\infty} (U(s)\varphi)\eta_t(ds)$$

where

$$e^{-tf(x)} = \int_0^{+\infty} e^{-xs}\eta_t(ds) \ (t > 0).$$

Thus

$$U^f(t)\varphi = \int_0^{+\infty} (\varphi * m_s)\eta_t(ds) = \varphi * m_t^f$$

where

$$m_t^f = \int_0^{+\infty} m_s \eta_t(ds) \tag{60}$$

is the Radon measure

$$\langle m_t^f, \zeta \rangle = \int_0^{+\infty} \langle m_s, \zeta \rangle \eta_t(ds), \ (\zeta \in C_c(\mathbb{R}^N)).$$

Let m_s be a function $p_s \in L^1_+(\mathbb{R}^N)$. Then $\langle m_s, \zeta \rangle = \int_{\mathbb{R}^N} p_s(y)\zeta(y)dy$ and

$$\begin{aligned} \langle m_t^f, \zeta \rangle &= \int_0^{+\infty} \left(\int_{\mathbb{R}^N} p_s(y)\zeta(y)dy \right) \eta_t(ds) \\ &= \int_{\mathbb{R}^N} \left(\int_0^{+\infty} p_s(y)\eta_t(ds) \right) \zeta(y)dy \end{aligned}$$

shows that

$$m_t^f(dy) == p_t^f(y)dy$$

where

$$p_t^f(y) := \int_0^{+\infty} p_s(y)\eta_t(ds)$$

is an L^1 function. Finally Theorem 25 ends the proof. ∎

We refer to Remark 22 to check how (12) could be satisfied by $\left(U^f(t)\right)_{t \geqslant 0}$. Since the heat semigroup is holomorphic in $L^1(\mathbb{R}^N)$ then so is $\left(U^f(t)\right)_{t \geq 0}$ for any Bernstein function f (see [21]) and then Theorem 27 implies:

Corollary 28 *Let $(U(t))_{t \geqslant 0}$ be the heat C_0-semigroup on $L^1(\mathbb{R}^N)$ and let f be a Bernstein function. Then $\left(U_V^f(t)\right)_{t \geq 0}$ is a compact C_0-semigroup if the sublevel sets Ω_M are "thin at infinity".*

We end this section with some usual examples covered by Corollary *28.* Note that x^α $(x > 0)$ for $0 < \alpha \leq 1$ is a Bernstein function f (see [31] Example 3.9.16, p. 180) and $\left\{U^f(t); t \geqslant 0\right\}$, the so-called symmetric stable semigroup of order 2α, corresponds to $F(\zeta) = |\zeta|^{2\alpha}$. Note that $\ln(1 + x)$ $(x > 0)$ is a Bernstein function (see [31] Example 3.9.15, p. 180) so that $\ln(1 + x^\alpha)$ $(x > 0)$ is also a Bernstein function f (see [31] Corollary 3.9.36, p. 206) and $\left\{U^f(t); t \geqslant 0\right\}$, the so-called geometric α-stable semigroup, corresponds to $F(\zeta) = \ln(1 + |\zeta|^\alpha)$ $(0 < \alpha \leq 2)$. Finally, $(x + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$ is a Bernstein function f (see [64]) and $\left\{U^f(t); t \geqslant 0\right\}$, the relativistic α-stable semigroup generated by (4), corresponds to $F(\zeta) = (|\zeta|^2 + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$.

Remark 29 *We can also deal with perturbation of generators of convolution semigroups by indefinite potentials, see Section 9 and Remark 64.*

5 Compactness results on $L^1(\Omega; d, \mu)$

In this section (and the following one) we complement the main compacteness results in Section 3 in L^1 spaces over *separable metric measure spaces* $(\Omega; d, \mathcal{A}, \mu)$ where (Ω, d) denotes a separable metric space, $\mathcal{A}$ is the σ-algebra of Borel subsets of Ω and μ is a σ-finite Borel measure on Ω. It follows that $\mathcal{A}$ is separable, i.e. is generated by a denumerable sub-family $\mathcal{D} \subset \mathcal{A}$ (see [55] Theorem 1.8, p. 5) and consequently (see e.g. [8] p. 98) $L^1(\Omega; A, \mu)$ is *separable.* We assume also that

$$\text{bounded Borel sets have finite } \mu\text{-measure.} \tag{61}$$

The existence of a metric d allows to understand further the key conditions (25)(26). Let $(U(t))_{t\geqslant 0}$ be a sub-stochastic C_0-semigroup on $L^1(\Omega; \mathcal{A}, \mu)$ with generator T. We complement Theorem 13 by:

Theorem 30 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). We denote by $(U_V(t))_{t\geqslant 0}$ the corresponding perturbed sub-stochastic C_0-semigroup. We assume that*

$$(1-T)^{-1} : L^1(\Omega; \mu) \to L^1(\Xi)$$

is weakly compact for any bounded Borel set Ξ. Let $G_1(x, y)$ be the kernel of $(1-T)^{-1}$. If

$$\lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} G_1(x, y)\mu(dx) = 0, \quad \forall M > 0 \tag{62}$$

(for some $x_0 \in \Omega$) then (48) holds (and then T_V is resolvent compact).

Proof. Note that (62) is x_0-independent. As noted in Remark 23, the existence of the kernel $G_1(x, y)$ follows from the separability of $L^1(\Omega; \mu)$ and the weak compactness assumption. We decompose $\chi_{\Omega_M}(1 - T_V)^{-1}$ as

$$\begin{aligned} \chi_{\Omega_M}(1-T)^{-1} &= \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1-T)^{-1} \\ &\quad + \chi_{\{x\in\Omega_M, d(x,x_0)<C\}}(1-T)^{-1}. \end{aligned}$$

By assumption, $\chi_{\{x\in\Omega_M, d(x,x_0)<C\}}(1 - T_V)^{-1}$ is weakly compact. On the other hand, the norm of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1-T)^{-1}$ is given by

$$\sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} G_1(x, y)\mu(dx)$$

so (by Assumption (62))

$$\left\|\chi_{\Omega_M}(1-T)^{-1}-\chi_{\{x\in\Omega_M,d(x,x_0)<C\}}(1-T)^{-1}\right\|_{\mathcal{L}(L^1(\Omega;\mu))}$$

is arbitrarily small for C large enough. Hence $\chi_{\Omega_M}(1-T_V)^{-1}$ is weakly compact. ■

We note that if $(U_V(t))_{t\geqslant 0}$ is operator norm continuous (e.g. if $(U(t))_{t\geqslant 0}$ is operator norm continuous and (44) is satisfied) then Theorem 30 implies the compactness of the C_0-semigroup $(U_V(t))_{t\geqslant 0}$. We have also:

Theorem 31 *Let (Ω,d,μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). We denote by $(U_V(t))_{t\geqslant 0}$ the corresponding perturbed sub-stochastic C_0-semigroup. We assume that (12) is satisfied. Let*

$$U(t):L^1(\Omega;\mu)\to L^1(\Xi)$$

be weakly compact for any bounded Borel set Ξ . Let $p_t(x,y)$ be the kernel of $U(t)$. If

$$\lim_{C\to+\infty}\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}}p_t(x,y)\mu(dx)=0\quad (t>0) \tag{63}$$

(for some $x_0\in\Omega$) then (49) holds (and then $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup).

Proof. Arguing as in the previous proof, we decompose $\chi_{\Omega_M}U(t)$ as

$$\begin{aligned}\chi_{\Omega_M}U(t) &= \chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}}U(t)\\ &\quad+\chi_{\{x\in\Omega_M,d(x,x_0)<C\}}U(t)\end{aligned}$$

Since $\chi_{\{x\in\Omega_M,d(x,x_0)<C\}}U(t)$ is weakly compact and

$$\left\|\chi_{\Omega_M}U(t)-\chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}}U(t)\right\|=\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}}p_t(x,y)\mu(dx)$$

goes to zero as $M\to+\infty$ then $\chi_{\Omega_M}U(t)$ is weakly compact, i.e. (49) holds. ■

We link now Theorem 30 and Theorem 31 to the notion of sublevels sets *"thin at infinity"*. We say that a Borel set $\Xi\subset\Omega$ is "thin at infinity" if there exists a point $\overline{y}\in\Omega$ such that for all $M>0$

$$\mu\{\Xi\cap B(y;M)\}\to 0 \ \text{ as } d(y,\overline{y})\to+\infty \tag{64}$$

where $B(y;M)$ is the ball centered at y with radius M. This definition is $\overline{y}$-independent.

We give first a basic preliminary result.

Lemma 32 *We assume that*

$$v(r) := \sup_{x\in\Omega} \mu(B(x,r)) < +\infty, \ \forall r \geqslant 0.$$

Let H be the integral operator

$$L^1(\Omega;\mu) \ni \varphi \to \int_\Omega h(x,y)\varphi(y)\mu(dy) \in L^1(\Omega;\mu)$$

satisfying a kernel estimate of the form

$$h(x,y) \leq f(d(x,y))$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and such that (for sufficiently large r)

$$r \to f(r)v(r+1)$$

is non increasing and integrable at infinity. Then:

(i) H is a bounded operator on $L^1(\Omega;\mu)$.

(ii) If a Borel set $\Xi \subset \Omega$ is "thin at infinity" in the sense (64) then

$$H : L^1(\Omega;\mu) \to L^1(\Xi;\mu) \text{ is weakly compact.}$$

Proof. (i) By domination, it suffices to show that

$$\varphi \in L^1(\Omega;\mu) \to \int f(d(x,y))\varphi(y)\mu(dy) \in L^1(\Omega;\mu) \tag{65}$$

is a bounded operator. This holds if and only if there exists $C > 0$ such that

$$\int f(d(x,y))\mu(dx) \leq C \quad \forall y \in \Omega.$$

We have

$$\begin{aligned}
\int f(d(x,y))\mu(dx) &= \int_{\{d(x,y)<1\}} f(d(x,y))\mu(dx) && (66)\\
&\quad + \sum_{n=1}^{\infty} \int_{\{n\leq d(x,y)<n+1\}} f(d(x,y))\mu(dx) \\
&\leq f(0)\mu(B(y,1)) + \sum_{n=1}^{\infty} f(n)\left[\mu(B(y,n+1)) - \mu(B(y,n))\right] \\
&= \left[f(0)-f(1)\right]\mu(B(y,1)) + \left[f(1)-f(2)\right]\mu(B(y,2)) + \cdots \\
&= \sum_{n=0}^{\infty} \left[f(n)-f(n+1)\right]\mu(B(y,n+1)) && (67)
\end{aligned}$$

which is finite if

$$\sum_{n=0}^{\infty} f(n)\mu(B(y,n+1)) < \infty, \quad \sum_{n=0}^{\infty} f(n+1)\mu(B(y,n+1)) < \infty$$

or

$$\sum_{n=0}^{\infty} f(n)v(n+1) < \infty, \quad \sum_{n=0}^{\infty} f(n+1)v(n+1) < \infty$$

or equivalently

$$\sum_{n=0}^{\infty} f(n)v(n+1) < \infty$$

(since $v(n) \leq v(n+1)$) which follows from $\int_1^{+\infty} f(r)v(r+1)dr < \infty$ and $r \to f(r)v(r+1)$ is nonincreasing.

(ii) We decompose the integral operator (65) by decomposing its kernel as

$$f(d(x,y)) = 1_{\Xi_c}(x) f(d(x,y)) + 1_{\widetilde{\Xi}_c}(x) f(d(x,y))$$

where

$$\Xi_c := \Xi \cap \{x; d(x,\overline{y}) \geqslant c\} \text{ and } \widetilde{\Xi}_c := \Xi \cap \{x; d(x,\overline{y}) < c\}$$

since $x \in \Xi$. Note that $f(d(x,y)) \leq f(0)$ so that

$$\varphi \in L^1(\Omega;\mu) \to \int 1_{\widetilde{\Xi}_c}(x) f(d(x,y))\varphi(y)u(dy) \in L^\infty(\widetilde{\Xi}_c;\mu)$$

and (since $\mu\left\{\widetilde{\Xi}_c\right\}$ is finite) the imbedding of $L^\infty(\widetilde{\Xi}_c;\mu)$ into $L^1(\widetilde{\Xi}_c;\mu)$ is weakly compact because a bounded subset of $L^\infty(\widetilde{\Xi}_c;\mu)$ is equi-integrable. It suffices to show that the norm of the second part goes to zero as $c \to +\infty$, i.e.

$$\sup_{y\in\Omega} \int_{\Xi\cap\{d(x,\overline{y})\geqslant c\}} f(d(x,y))\mu(dx) \to 0 \quad \text{as } c \to +\infty.$$

Consider first the integral

$$\begin{aligned} & \int_{\Xi\cap\{d(x,\overline{y})\geqslant c\}} f(d(x,y))\mu(dx) \\ = & \sum_{n=0}^{\infty} \int_{\{n\leq d(x,y)<n+1\}\cap\Xi\cap\{d(x,\overline{y})\geqslant c\}} f(d(x,y))\mu(dx) \\ \leq & \sum_{n=0}^{\infty} f(n)\mu\left[\{n \leq d(x,y) < n+1\} \cap \Xi \cap \{d(x,\overline{y}) \geqslant c\}\right]. \end{aligned}$$

We note that

$$\begin{aligned}
& \sum_{n=m}^{\infty} f(n)\mu\left[\{n \leq d(x,y) < n+1\} \cap \Xi \cap \{d(x,\overline{y}) \geqslant c\}\right] \\
\leq & \sum_{n=m}^{\infty} f(n)\mu\left[\{n \leq d(x,y) < n+1\}\right] \\
= & \sum_{n=m}^{\infty} f(n)\left[\mu(B(y,n+1)) - \mu(B(y,n))\right] \\
\leq & \sum_{n=m}^{\infty} f(n)\left[\mu(B(y,n+1)) + \mu(B(y,n))\right] \\
\leq & \; c\sum_{n=m}^{\infty} f(n)\left[v(n+1) + v(n)\right]
\end{aligned}$$

so that, for any $\varepsilon > 0$ there exists an integer $\overline{m}$ such that

$$\sum_{n=\overline{m}}^{\infty} f(n)\mu\left[\{n \leq d(x,y) < n+1\} \cap \Xi \cap \{d(x,\overline{y}) \geqslant c\}\right] \leq \varepsilon \text{ uniformly in } y \in \Omega.$$

It suffices to show that

$$\sum_{n=0}^{\overline{m}} f(n)\mu\left[\{n \leq d(x,y) < n+1\} \cap \Xi \cap \{d(x,\overline{y}) \geqslant c\}\right] \to 0 \quad \text{as } c \to +\infty$$

uniformly in $y \in \Omega$, or equivalently for *any* $n \leq \overline{m}$

$$\mu\left[\{n \leq d(x,y) < n+1\} \cap \Xi \cap \{d(x,\overline{y}) \geqslant c\}\right] \to 0 \quad \text{as } c \to +\infty \tag{68}$$

uniformly in $y \in \Omega$. The inequality

$$d(y,\overline{y}) \geqslant |d(x,\overline{y}) - d(x,y)| \geqslant c - (n+1)$$

for $c > (n+1)$ shows that either the set

$$\{x; n \leq d(x,y) < n+1\} \cap \{x; d(x,\overline{y}) \geqslant c\}$$

is *empty* (and then $\mu\left[\{n \leq d(x,y) < n+1\} \cap \Xi \cap \{d(x,\overline{y}) \geqslant c\}\right] = 0$) or $d(y,\overline{y}) \geqslant c - (n+1)$. On the other hand, by assumption, for any n

$$\mu\left[\{x; d(x,y) < n+1\} \cap \Xi\right] \to 0 \quad \text{as } d(y,\overline{y}) \to \infty$$

and then (68) follows. ■

Now Theorem 30, Theorem 31 and Lemma 32 imply:

Theorem 33 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). We denote by $(U_V(t))_{t\geqslant 0}$ the corresponding perturbed sub-stochastic C_0-semigroup. We assume that the sublevel sets Ω_M are "thin at infinity" in the sense (64).*

(i) If the kernel $G(x,y)$ of $(1-T)^{-1}$ satisfies an estimate of the form $G(x,y) \leq f(d(x,y))$ where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and such that (for large r) $r \to f(r)v(r+1)$ is nonincreasing and integrable at infinity then T_V is resolvent compact.

(ii) Let (12) be satisfied. If for each $t > 0$, the kernel $p_t(.,.)$ of $U(t)$ satisfies an estimate of the form $p_t(.,.) \leq f_t(d(x,y))$ where $f_t : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and such that (for large r) $r \to f_t(r)v(r+1)$ is nonincreasing and integrable at infinity then $(U_V(t))_{t\geqslant 0}$ is a compact C_0-semigroup.

Remark 34 *Theorem 33 applies to the different examples of kernel estimates (21)(22)(23) arising in the theory of Markov process*

$$f_t(r) := \frac{C}{t^\gamma}\exp\left(-\frac{r^2}{Ct}\right), \quad \frac{C}{t^{\frac{\alpha}{\beta}}}\exp\left(-\frac{r^{\frac{\beta}{\beta-1}}}{C^{\frac{\beta}{\beta-1}}t^{\frac{\beta}{\beta-1}}}\right) \quad or \quad \frac{C}{t^{\frac{\alpha}{\beta}}}\left(1+\frac{r}{t^{\frac{1}{\beta}}}\right)^{-(\alpha+\beta)},$$

provided we impose an appropriate volume growth

$$r \to v(r)$$

in order to meet the above conditions on $r \to f_t(r)v(r+1)$.

6 Spectral gaps on $L^1(\Omega; d, \mu)$

We recall that $s(T_V) \in \sigma(T_V)$ and $s(T_V)$ is equal to the type of $(U_V(t))_{t\geqslant 0}$. Note that $s(T_V) \leq 0$ by the contraction of $(U_V(t))_{t\geqslant 0}$. We recall also that the spectral gap (or essential compactness) property of perturbed C_0-semigroups $(U_V(t))_{t\geqslant 0}$ refers to the strict inequality

$$\omega_{ess}(U_V) < s(T_V)$$

while a spectral gap property of perturbed generator T_V refers to

$$s_{ess}(T_V) < s(T_V). \tag{69}$$

We deal first with perturbed generators.

Theorem 35 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). We denote by $(U_V(t))_{t\geqslant 0}$ the corresponding perturbed sub-stochastic C_0-semigroup. Let*

$$(1-T)^{-1} : L^1(\Omega) \to L^1(\Xi)$$

be weakly compact for any bounded Borel set Ξ. We assume that the kernel $G_1(x,y)$ of $(1-T)^{-1}$ satisfies the estimate

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} G_1(x,y)\mu(dx) < \frac{1}{1-s(T_V)} \tag{70}$$

(for some $x_0 \in \Omega$). Then

$$s(T_V) - s_{ess}(T_V) \geq \widehat{\eta}$$

where

$$\widehat{\eta} := \frac{\eta((1-s(T_V))}{r_{ess}\left[(1-T_V)^{-1}\right]}$$

and η is difference between the right and left hand sides of (70).

Proof. We choose an arbitrary $\varepsilon > 0$ such that

$$\varepsilon < \eta.$$

It is known (see e.g. [52] Proposition 2.5, p 67), for any $\beta \in \rho(T_V)$,

$$r_\sigma\left[(\beta - T_V)^{-1}\right] = \frac{1}{dist(\beta, \sigma(T_V))}, \tag{71}$$

in particular

$$r_\sigma\left((1-T_V)^{-1}\right) = \frac{1}{1-s(T_V)}$$

(since $s(T_V) \in \sigma(T_V)$) and

$$1 - s(T_V) = \frac{1}{r_\sigma\left((1-T_V)^{-1}\right)}.$$

Let

$$\lambda \in \sigma(T_V)$$

be an *arbitrary* spectral value of T_V and let

$$q := \operatorname{Im} \lambda$$

be its imaginary part. Note that $\operatorname{Re}\lambda \leq s(T_V)$.

Note the *uniform* domination in $q \in \mathbb{R}$

$$\begin{aligned} \left|(1+iq-T_V)^{-1}f\right| &= \left|\int_0^{+\infty} e^{-(1+iq)t}e^{-tT_V}f\,dt\right| \\ &\leq \int_0^{+\infty} e^{-t}e^{-tT_V}\,|f|\,dt = (1-T_V)^{-1}\,|f|\,. \end{aligned}$$

The same argument shows that

$$\left|(1+iq-T_V)^{-n}f\right| \leq (1-T_V)^{-n}\,|f|$$

for any integer n so that taking the $\frac{1}{n}$-powers of the operator norms and passing to the limit as $n \to \infty$

$$r_\sigma((1+iq-T_V)^{-1}) \leq r_\sigma((1-T_V)^{-1}) \quad \forall q \in \mathbb{R}. \tag{72}$$

We decompose $(1+iq-T_V)^{-1}$ as

$$\begin{aligned} (1+iq-T_V)^{-1} &= \chi_{\Omega_M^c}(1+iq-T_V)^{-1} + \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1+iq-T_V)^{-1} \\ &\quad +\chi_{\{x\in\Omega_M, d(x,x_0)<C\}}(1+iq-T_V)^{-1} \end{aligned} \tag{73}$$

where Ω_M^c is the complement of the sublevel set Ω_M. Since

$$\chi_{\{x\in\Omega_M, d(x,x_0)<C\}}(1+iq-T_V)^{-1}$$

is dominated by $\chi_{\{x\in\Omega_M, d(x,x_0)<C\}}(1-T_V)^{-1}$ which is itself dominated by

$$\chi_{\{x\in\Omega_M, d(x,x_0)<C\}}(1-T)^{-1}$$

then, by our assumption, the third operator in (73) is weakly compact. Moreover, we saw in the proof of Theorem 13 that the norm of $\chi_{\Omega_M^c}(1-T_V)^{-1}$ goes to zero as $M \to +\infty$ so that, by domination, the norm of $\chi_{\Omega_M^c}(1+iq-T_V)^{-1}$ goes to zero (*uniformly in* q) as $M \to +\infty$. Finally, the norm of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1+iq-T_V)^{-1}$ is less than or equal to that of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1-T_V)^{-1}$ which is itself less than or equal to the norm of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1-T)^{-1}$ i.e.

$$\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} G_1(x,y)\mu(dx).$$

It follows that for M and C large enough

$$\left\|\chi_{\Omega_M^c}(1+iq-T_V)^{-1} + \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}}(1+iq-T_V)^{-1}\right\| \leq \frac{1}{1-s(T_V)} - \varepsilon$$

uniformly in q. In L^1 spaces, the essential spectrum is stable by weakly compact perturbations (see [36], Proposition 2.c.10, p. 79) so that for M and C large enough

$$\begin{aligned} r_{ess}\left[(1+iq-T_V)^{-1}\right] &= r_{ess}\left[(\chi_{\Omega_M^c}+\chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}})(1+iq-T_V)^{-1}\right] \\ &\le \left\|(\chi_{\Omega_M^c}+\chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}})(1+iq-T_V)^{-1}\right\| \\ &< \frac{1}{1-s(T_V)}-\varepsilon = r_\sigma\left[(1-T_V)^{-1}\right]-\varepsilon \end{aligned}$$

uniformly in q so

$$\frac{1}{1-s(T_V)}-r_{ess}\left[(1+iq-T_V)^{-1}\right] > \varepsilon \text{ uniformly in } q. \tag{74}$$

By using (74) and (72) we get

$$\begin{aligned} &\frac{1}{r_{ess}\left[(1+iq-T_V)^{-1}\right]}-(1-s(T_V)) \\ =\ &\frac{\frac{1}{1-s(T_V)}-r_{ess}\left[(1+iq-T_V)^{-1}\right]}{\frac{1}{1-s(T_V)}r_{ess}\left[(1+iq-T_V)^{-1}\right]} \ge \frac{\varepsilon((1-s(T_V))}{r_{ess}\left[(1-T_V)^{-1}\right]} \end{aligned} \tag{75}$$

uniformly in q.

On the other hand, γ is an isolated eigenvalue of T_V with finite algebraic multiplicity if and only if $\frac{1}{1+iq-\gamma}$ is an isolated eigenvalue of $(1+iq-T_V)^{-1}$ with finite algebraic multiplicity, then any spectral value γ of T_V such that

$$\frac{1}{|1+iq-\gamma|} > r_{ess}\left[(1+iq-T_V)^{-1}\right]$$

is an isolated eigenvalue of T with finite algebraic multiplicity. Hence any spectral value λ of T_V (with imaginary part q) such that

$$|1-\operatorname{Re}\lambda| < \frac{1}{r_{ess}\left[(1+iq-T_V)^{-1}\right]}$$

is an isolated eigenvalue of T with finite algebraic multiplicity. Since

$$0 \le 1-\operatorname{Re}\lambda = (1-s(T_V)\ )+(s(T_V)-\operatorname{Re}\lambda)$$

then any spectral value λ of T_V (with imaginary part q) such that

$$s(T_V)-\operatorname{Re}\lambda < \frac{1}{r_{ess}\left[(1+iq-T_V)^{-1}\right]}-(1-s(T_V))$$

is an isolated eigenvalue of T with finite algebraic multiplicity. Finally, (75) show that any spectral value λ of T_V (with imaginary part q) such that

$$s(T_V) - \operatorname{Re}\lambda < \frac{\varepsilon((1 - s(T_V))}{r_{ess}\left[(1 - T_V)^{-1}\right]}$$

is an isolated eigenvalue of T with finite algebraic multiplicity. The arbitrariness of $\varepsilon < \eta$ ends the proof. ■

We have a better insight into (70) if we note the estimates

$$\sup_{y \in \Omega} \int G_1(x,y)\mu(dx) = \left\|(1-T)^{-1}\right\| \geq r_\sigma\left((1-T)^{-1}\right) = \frac{1}{1-s(T)} \geq \frac{1}{1-s(T_V)}.$$

Theorem 36 *Let the conditions of Theorem 35 be satisfied. If* $(U_V(t))_{t\geqslant 0}$ *is operator norm continuous then* $\omega_{ess}(U_V) < s(T_V)$ *i.e.* $(U_V(t))_{t\geqslant 0}$ *has a spectral gap.*

Proof. By the operator norm continuity of $(U_V(t))_{t\geqslant 0}$,

$$(\lambda - T_V)^{-1} = \int_0^{+\infty} e^{-\lambda t} U_V(t) dt \quad (\operatorname{Re}\lambda > s(T_V))$$

is given by a *Bochner* integral (instead of simply a strong integral) so that Riemann-Lebesgue Lemma holds

$$\left\|(\lambda - T_V)^{-1}\right\| \to 0 \text{ as } |\operatorname{Im}\lambda| \to \infty. \tag{76}$$

By Theorem 35, there exists $\alpha > 0$ such that

$$\sigma(T_V) \cap \{s(T_V) - \alpha \leqslant \operatorname{Re}\lambda \leq s(T_V)\}$$

consists of a (non-empty) set isolated eigenvalues with finite algebraic multiplicities. This set must be finite. Indeed, otherwise we would have a sequence of eigenvalues $\nu_k = \alpha_k + i\beta_k$ such that $\alpha_k \in [s(T_V) - \alpha,\ s(T_V)]$ and $|\beta_k| \to \infty$ with normalized eigenvectors x_k. Without loss of generality, we may assume that

$$\alpha_k \to \alpha \leq s(T_V) \leq 0.$$

Since $T_V x_k = (\alpha_k + i\beta_k)x_k$, i.e.

$$(1 + i\beta_k - T_V)x_k = (1 - \alpha_k)x_k$$

then

$$\begin{aligned} 1 &= \|x_k\| = |(1-\alpha_k)| \left\|(1 + i\beta_k - T_V)^{-1} x_k\right\| \\ &\leq |(1-\alpha_k)| \left\|(1 + i\beta_k - T_V)^{-1}\right\| \end{aligned}$$

which is impossible if $|\beta_k| \to \infty$ because of (76).

We denote by $\{\nu_1, ..., \nu_J\}$ this finite set of eigenvalues. Let P be the (finite dimensional) spectral projection corresponding to this finite set of eigenvalues. Note that this projection commutes with $U_V(t)$. We denote by Y its finite dimensional range. We decompose $L^1(\Omega)$ as

$$L^1(\Omega) = X \oplus Y$$

where $X = (I - P)(L^1(\Omega))$. Then

$$\sigma(T_V) = \{\nu_1, ..., \nu_J\} \cup \sigma(T_{V|X})$$

where $T_{V|X}$ is the restriction of T_V to X (with domain $D(T_V) \cap X$) and

$$\sigma(T_{V|X}) = \sigma(T_V) \cap \{\mathrm{Re}\, \lambda < s(T_V) - \alpha\}.$$

We decompose then $U_V(t)$ as

$$U_V(t) = U_V(t)P + U_V(t)(I - P).$$

It follows that

$$\sigma_{ess}(U_V(t)) = \sigma_{ess}(U_V(t)(I - P)) \subset \sigma(U_V(t)(I - P))$$

where $(U_V(t)(I - P))_{t \geq 0}$ is identified to the C_0-semigroup on X with generator $T_{V|X}$. Thus

$$e^{\omega_{ess} t} = r_{ess}(U_V(t)) \leq r_\sigma(U_V(t)(I - P)).$$

Since $(U_V(t)(I - P))_{t \geq 0}$ is also operator norm continuous then the spectral mapping theorem

$$\sigma(U_V(t)(I - P)) - \{0\} = e^{t\sigma(T_{V|X})}$$

holds (see e.g. [52] p 87) so that $r_\sigma(U_V(t)(I - P)) \leq e^{(s(T_V) - \alpha)t}$ and finally $\omega_{ess} < s(T_V)$. ■

We give now a second approach to spectral gaps for perturbed C_0-semigroups based on the weak type estimate (35).

Theorem 37 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). We denote by $(U_V(t))_{t \geqslant 0}$ the corresponding perturbed sub-stochastic C_0-semigroup. We assume that (12) is satisfied. Let $t > 0$ be fixed and let*

$$U(t) : L^1(\Omega) \to L^1(\Xi)$$

be weakly compact for any bounded Borel set Ξ. We assume that the kernel $p_t(x,y)$ of $U(t)$ satisfies the estimate

$$\sup_{M>0}\lim_{C\to+\infty}\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < e^{s(T_V)t} \tag{77}$$

for some $t>0$ (and some $x_0\in\Omega$). Then $\omega_{ess}(U_V) < s(T_V)$.

Proof. We denote by Ω_M^c the complement of Ω_M and decompose $U_V(t)$ as

$$\begin{aligned} U_V(t) &= \chi_{\Omega_M^c}U_V(t) + \chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}}U_V(t) \\ &\quad +\chi_{\{x\in\Omega_M,d(x,x_0)<C\}}U_V(t). \end{aligned} \tag{78}$$

Since $\chi_{\{x\in\Omega_M,d(x,x_0)<C\}}U_V(t)$ is dominated by $\chi_{\{x\in\Omega_M,d(x,x_0)<C\}}U(t)$ then, by our assumption, the third operator in (78) is weakly compact. Then the stability of the essential spectrum by weakly compact perturbations in L^1 spaces (see [36], Proposition 2.c.10, p. 79) shows that

$$\begin{aligned} e^{\omega_{ess}(U_V)t} &= r_{ess}\left[U_V(t)\right] = r_{ess}\left[(\chi_{\Omega_M^c} + \chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}})U_V(t)\right] \\ &\leq \left\|(\chi_{\Omega_M^c} + \chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}})U_V(t)\right\| \quad \forall M,C \end{aligned}$$

and then

$$\begin{aligned} e^{\omega_{ess}(U_V)t} &\leq \lim_{M\to\infty}\lim_{C\to\infty}\left\|(\chi_{\Omega_M^c} + \chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}})U_V(t)\right\| \\ &= \lim_{M\to\infty}\lim_{C\to\infty}\left\|\chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}}U_V(t)\right\| \end{aligned}$$

since (35) shows that the norm of $\chi_{\Omega_M^c}U_V(t)$ goes to zero as $M\to+\infty$. On the other hand, the norm of $\chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}}U_V(t)$ is less than or equal to that of $\chi_{\{x\in\Omega_M,d(x,x_0)\geqslant C\}}U(t)$ i.e.

$$\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx)$$

so

$$\begin{aligned} e^{\omega_{ess}(U_V)t} &\leq \lim_{M\to\infty}\lim_{C\to\infty}\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) \\ &= \sup_{M>0}\lim_{C\to\infty}\sup_{y\in\Omega}\int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < e^{s(T_V)t} \end{aligned}$$

and $\omega_{ess}(U_V) < s(T_V)$. ■

Remark 38 *Actualy the proof of Theorem 37 provides the "quantitative" estimate*

$$\omega_{ess} \leq \inf_{t>0} \frac{1}{t} \ln \left(\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) \right).$$

The proof of Theorem 37 suggests an interesting variant.

Corollary 39 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). We denote by $(U_V(t))_{t\geqslant 0}$ the corresponding perturbed sub-stochastic C_0-semigroup. We assume that (12) is satisfied. Let $t > 0$ be fixed and let*

$$U(t) : L^1(\Omega) \to L^1(\Xi)$$

be weakly compact for any bounded Borel set Ξ. We assume that the kernel $p_t(x,y)$ of $U(t)$ satisfies the estimate

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < e^{s(T)t} \tag{79}$$

(for some $x_0 \in \Omega$) where $s(T)$ be the spectral bound of T. Then either $s(T_V) < s(T)$ or $s(T_V) = s(T)$ and $\omega_{ess}(U_V) < s(T_V)$.

Proof. Since $s(T_V) \leq s(T)$ then either $s(T_V) < s(T)$ or $s(T_V) = s(T)$ and then we can of course replace $s(T)$ by $s(T_V)$ in (79) and appeal to Theorem 37. ■

In particular, if $(U(t))_{t\geqslant 0}$ is a stochastic C_0-semigroup (i.e. mass preserving on the positive cone) then $\int p_t(x,y)\mu(dx) = 1$ and $s(T) = 0$ so that we have:

Corollary 40 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V be satisfy (6)(8). Let $(U(t))_{t\geqslant 0}$ be a stochastic C_0-semigroup (i.e. mass preserving on the positive cone). We assume that (12) is satisfied. Let $t > 0$ be fixed and let*

$$U(t) : L^1(\Omega) \to L^1(\Xi)$$

be weakly compact for any bounded Borel set Ξ. If the kernel $p_t(x,y)$ of $U(t)$ satisfies the estimate

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < 1 \tag{80}$$

(for some $x_0 \in \Omega$) then either $s(T_V) < 0$ or $\omega_{ess}(U_V) < s(T_V) = 0$.

We consider now the case where $(U(t))_{t\geqslant 0}$ is a *sub-Markov* C_0-semigroup, i.e. acts in all L^p spaces as a positive contraction C_0-semigroup. We denote it by $(U^p(t))_{t\geqslant 0}$ as a C_0-semigroup acting on $L^p(\Omega;\mu)$ with generator T^p. We denote by $\left(U_V^p(t)\right)_{t\geq 0}$ the corresponding perturbed C_0-semigroup in $L^p(\Omega;\mu)$ and by T_V^p its generator. Let $s(T_V^p)$ be the spectral bound of T_V^p. Finally, let $\omega_{ess}(U_V^p)$ be the essential type of $\left(U_V^p(t)\right)_{t\geq 0}$.

Theorem 41 *Let (Ω, d, μ) be a separable metric measure space satisfying (61). Let V satisfy (6)(8). Let $(U(t))_{t\geqslant 0}$ be a sub-Markov C_0-semigroup. We assume that (12) is satisfied. Let $t>0$ be fixed and let*

$$U(t): L^1(\Omega) \rightarrow L^1(\Xi)$$

be compact for any bounded Borel set Ξ. If the kernel $p_t(x,y)$ of $U(t)$ satisfies the estimate

$$\sup_{M>0} \lim_{C\rightarrow +\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < e^{ps(T_V^p)t}$$

(for some $x_0\in\Omega$) then $\omega_{ess}(U_V^p) < s(T_V^p)$.

Proof. We recall that $s(T_V^p)$ is equal to the *type* of $\left(U_V^p(t)\right)_{t\geq 0}$ [74]. We decompose $U_V^p(t)$ as

$$\begin{aligned} U_V^p(t) &= \chi_{\Omega_M^c} U_V^p(t) + \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}} U_V^p(t) \\ &\quad + \chi_{\{x\in\Omega_M, d(x,x_0)< C\}} U_V^p(t) \end{aligned}$$

where Ω_M^c is the complement of the sublevel set Ω_M. We note the compactness of $\chi_{\{x\in\Omega_M, d(x,x_0)<C\}} U^p(t)$ in $L^p(\Omega)$ (by interpolation from the L^1 compactness assumption) and then the domination

$$\chi_{\{x\in\Omega_M, d(x,x_0)<C\}} U_V^p(t) \leq \chi_{\{x\in\Omega_M, d(x,x_0)<C\}} U^p(t)$$

shows that $\chi_{\{x\in\Omega_M, d(x,x_0)<C\}} U_V^p(t)$ is compact in $L^p(\Omega)$ by Doods-Fremlin's theorem (see e.g. [1] Theorem 5.20, p. 286). Moreover, by (35) the L^1-operator norm of $\chi_{\Omega_M^c} U_V(t)$ goes to zero as $M\rightarrow +\infty$ while its L^∞-operator norm is less than or equal to one. Then, by Riesz-Thorin interpolation theorem, the L^p-operator norm of $\chi_{\Omega_M^c} U_V^p(t)$ goes also to zero as $M\rightarrow +\infty$. Finally, the L^1-operator norm of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}} U_V(t)$ is less than or equal to that of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}} U(t)$ i.e.

$$\sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx)$$

(and its L^∞-operator norm is less than or equal to one) so that, by Riesz-Thorin interpolation theorem, the L^p-operator norm of $\chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}} U_V^p(t)$ is less than or equal to

$$(\sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx))^{\frac{1}{p}}.$$

It follows that for M and C large enough the L^p-operator norm of $\chi_{\Omega_M^c} U_V^p(t) + \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}} U_V^p(t)$ is less than $(e^{ps(T_V^p)t})^{\frac{1}{p}} = e^{s(T_V^p)t}$. Then the stability of the essential spectrum by compact perturbations shows that

$$\begin{aligned} e^{\omega_{ess}(U_V^p)t} &= r_{ess}\left[U_V^p(t)\right] = r_{ess}\left[(\chi_{\Omega_M^c} + \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}})U_V^p(t)\right] \\ &\leq \left\|(\chi_{\Omega_M^c} + \chi_{\{x\in\Omega_M, d(x,x_0)\geqslant C\}})U_V^p(t)\right\| < e^{s(T_V^p)t} \end{aligned}$$

so that $\omega_{ess}(U_V^p) < s(T_V^p)$. ■

Remark 42 *In Theorem 37, if we replace (77) by*

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) < e^{\alpha t}$$

for some $\alpha \leq s(T_V)$ then the proof above gives $\omega_{ess}(U_V) < \alpha$. This formulation of Theorem 37 will be used in the proof of Theorem 58 below. More generally, the proof of Theorem 37 shows the "quantitative" estimate

$$\omega_{ess}(U_V) \leq \inf_{t>0} \frac{1}{t} \ln\left(\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\Omega} \int_{\{x\in\Omega_M;\ d(x,x_0)\geqslant C\}} p_t(x,y)\mu(dx) \right).$$

Remark 43 *Note that if the C_0-semigroup $\{U_V(t); t \geqslant 0\}$ is irreducible and essentially compact (i.e. $\omega_{ess}(U_V) < s(T_V)$) then $s(T_V)$ is a strictly dominant (algebraically simple) eigenvalue of T_V and*

$$e^{-s(T_V)t}U_V(t) \to P \quad \text{as } t \to +\infty$$

in operator norm where P is the one-dimensional spectral projection associated to the leading eigenvalue $s(T_V)$ (see e.g. [52] p. 343-344); in the case $s(T_V) = 0$, we have the so-called "exponential return to equilibrium". Besides weighted Schrödinger operators (see Theorem 53 below), this occurs e.g. in neutron transport theory [47].

7 On weighted Laplacians

Let $h \in C^2(\mathbb{R}^N)$ such that $h(x) > 0 \ \forall x \in \mathbb{R}^N$ and let $\mu(dx) = h^2(x)dx$. We define the weighted Laplacian on $L^2(\mathbb{R}^N; \mu(dx))$

$$\triangle^\mu := \frac{1}{h^2}\operatorname{div}(h^2\nabla) = \triangle + 2\frac{\nabla h.\nabla}{h}.$$

This is (minus) the self-adjoint operator in $L^2(\mathbb{R}^N; \mu(dx))$ associated to the Dirichlet form

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \, \mu(dx)$$

on

$$H^1(\mathbb{R}^N; \mu) = \left\{\varphi \in L^2(\mathbb{R}^N; \mu), \ \frac{\partial\varphi}{\partial x_i} \in L^2(\mathbb{R}^N; \mu), \ 1 \le i \le N\right\}$$

(see e.g. [11] Section 4.7, [22]). Let $V := \frac{\triangle h}{h}$. It is easy to see that

$$\begin{aligned}
\triangle^\mu\varphi &= \triangle\varphi + 2\frac{\nabla h.\nabla\varphi}{h} = \frac{1}{h}\left[h\triangle\varphi + 2\nabla h.\nabla\varphi + \varphi\triangle h - V\varphi h\right] \\
&= \frac{1}{h}\left[\triangle\varphi h - V\varphi h\right]
\end{aligned}$$

i.e.

$$\triangle^\mu = \frac{1}{h} \circ (\triangle - V) \circ h.$$

Thus the weighted Laplacian $\triangle^\mu$ in $L^2(\mathbb{R}^N; \mu(dx))$ is unitarily equivalent to the Schrödinger operator $\triangle - \frac{\triangle h}{h}$ on $L^2(\mathbb{R}^N; dx)$ by the unitary transformation

$$I : \varphi \in L^2(\mathbb{R}^N; \mu(dx)) \to h\varphi \in L^2(\mathbb{R}^N; dx).$$

This shows that the weighted Laplacian $\triangle^\mu$ in $L^2(\mathbb{R}^N; \mu(dx))$ has the same spectral properties (i.e. resolvent compactness, spectral gaps...) as the Schrödinger operator $\triangle - \frac{\triangle h}{h}$ on $L^2(\mathbb{R}^N; dx)$. We begin with several *compactness results* for weighted Laplacians related to thinness properties of sublevels sets of V. We start with the following result already obtained in [40] by other means.

Proposition 44 *Let $h \in C^2(\mathbb{R}^N)$ with $h(x) > 0 \ \forall x \in \mathbb{R}^N$. We assume that $\frac{\triangle h}{h}$ is bounded from below. Then the weighted Laplacian $\triangle^\mu$ generates a compact C_0-semigroup on $L^2(\mathbb{R}^N; \mu(dx))$ provided that the sublevel sets Ω_M of $\frac{\triangle h}{h}$ are "thin at infinity".*

Proof. Let $V := \frac{\triangle h}{h}$. Up to a bounded perturbation, without loss of generality, we can assume that $V \geqslant 0$. Then "$\triangle - V$", or more rigorously $\triangle_V$, generates a compact C_0-semigroup on $L^1(\mathbb{R}^N; dx)$ (see Theorem 25) and in $L^2(\mathbb{R}^N; dx)$ by an interpolation argument. We conclude by a similarity argument. ∎

Remark 45 *It follows from Proposition 44 that the imbedding of $H^1(\mathbb{R}^N; \mu)$ into $L^2(\mathbb{R}^N; \mu)$ is compact if $\frac{\triangle h}{h}$ is bounded from below and its sublevel sets are "thin at infinity"; see also [19].*

Generally, the function h is written in the form $h(x) := e^{-\frac{\Phi}{2}(x)}$ where Φ is a real C^2 function on $\mathbb{R}^N$, i.e.

$$\mu(dx) = e^{-\Phi(x)} dx.$$

Note that in this case

$$\triangle^{\mu} = \triangle + 2\frac{\nabla h.\nabla}{h} = \triangle - \nabla\Phi.\nabla$$

in $L^2(\mathbb{R}^N; e^{-\Phi(x)}dx)$; we do *not* assume a priori that $e^{-\Phi(x)}$ is integrable. It is known that

$$V := \frac{\triangle h}{h} = \frac{1}{4}\left|\nabla\Phi(x)\right|^2 - \frac{1}{2}\triangle\Phi(x).$$

The (minus) Schrödinger operators

$$\triangle_{\Phi} := -\triangle + \frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi$$

in $L^2(\mathbb{R}^N; dx)$ are also known as the Witten Laplacians (on 0-forms) and were studied in particular in [27] in connection with Fokker-Planck operators. Thus Proposition 44 takes the form:

Corollary 46 *Let Φ be a real C^2 function on $\mathbb{R}^N$. If $\frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi$ is bounded from below then the weighted Laplacian $\triangle^{\mu}$ on $L^2(\mathbb{R}^N; e^{-\Phi(x)}dx)$ generates a compact C_0-semigroup provided that the sublevel sets of $\frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi$ are "thin at infinity".*

Remark 47 *The Ornstein-Uhlenbeck generator $\triangle - x.\nabla$ is a weighted Laplacian in $L^2(\mathbb{R}^N; e^{-\frac{|x|^2}{2}}dx)$ unitarily equivalent to (minus) $-\triangle + \frac{|x|^2}{4} - \frac{N}{2}$ (the harmonic oscillator) in $L^2(\mathbb{R}^N; dx)$ and is known to generate a compact C_0-semigroup. We point out that the Ornstein-Uhlenbeck C_0-semigroup is not compact in $L^1(\mathbb{R}^N; e^{-\frac{|x|^2}{2}}dx)$ (see [11] Section 4.3) while the C_0-semigroup generated by (minus) the harmonic oscillator is compact in $L^1(\mathbb{R}^N; dx)$.*

We revisit now various examples considered in the literature *in L^2 setting*. The following potential appears e.g. in [25][32]

$$\Phi(x)=\frac{1}{h}\sum_{j=1}^{N}(\frac{\lambda}{12}x_j^4+\frac{\nu}{2}x_j^2)+\frac{1}{h}\frac{I}{2}\sum_{j=1}^{N}\left|x_j-x_{j+1}\right|^2 \tag{81}$$

(with the convention $x_{N+1}=x_1$) where $h>0,\ \lambda>0,\ \nu<0,\ I>0$.

Corollary 48 *Let Φ be of the form (81). Then $-\triangle_\Phi$ generates a (holomorphic) compact C_0-semigroup in $L^1(\mathbb{R}^N;dx)$.*

Proof. Writing (81) in the form

$$\Phi(x)=\alpha\sum_{j=1}^{N}x_j^4-\beta\sum_{j=1}^{N}x_j^2+\gamma\sum_{j=1}^{N}\left|x_j-x_{j+1}\right|^2$$

where $\alpha>0,\ \beta>0,\ \gamma>0$, it is easy to see that

$$\triangle\Phi=12\alpha\left|x\right|^2+\gamma(4-2\beta)N.$$

On the other (see [32]) there exists $c>0$ such that $\nabla\Phi(x).x\geqslant c\left|x\right|^4$ for $|x|$ large enough. Thus $\nabla\Phi(x).\frac{x}{|x|}\geqslant c\left|x\right|^3$ and then $\left|\nabla\Phi(x)\right|\geqslant c\left|x\right|^3$ for $|x|$ large enough. Finally

$$\frac{1}{4}\left|\nabla\Phi\right|^2-\frac{1}{2}\triangle\Phi\geqslant\frac{c^2\left|x\right|^6}{4}-6\alpha\left|x\right|^2+\gamma(2-\beta)N\to+\infty$$

as $|x|\to+\infty$ and we are done. ■

Sometimes Φ enjoys useful decompositions. We give a result in this direction and then apply it to uniformly strictly convex Φ.

Corollary 49 *Let $\Phi=\Phi_1+\Phi_2$ where Φ_1,Φ_2 be C^2 functions such that $(\frac{\left|\nabla\Phi_1\right|^2}{4}-\frac{1}{2}\triangle\Phi_1)+\frac{1}{2}\nabla\Phi_1.\nabla\Phi_2$ and $\frac{\left|\nabla\Phi_2\right|^2}{4}-\frac{1}{2}\triangle\Phi_2$ are bounded from below. If the sublevel sets of $\frac{\left|\nabla\Phi_2\right|^2}{4}-\frac{1}{2}\triangle\Phi_2$ are "thin at infinity" then $-\triangle_\Phi$ generates a (holomorphic) compact C_0-semigroup in $L^1(\mathbb{R}^N;dx)$.*

Proof. We note that

$$\triangle_\Phi:=-\triangle+(\frac{\left|\nabla\Phi_1\right|^2}{4}-\frac{1}{2}\triangle\Phi_1)+(\frac{\left|\nabla\Phi_2\right|^2}{4}-\frac{1}{2}\triangle\Phi_2)+\frac{1}{2}\nabla\Phi_1.\nabla\Phi_2.$$

We may assume that $(\frac{|\nabla\Phi_1|^2}{4} - \frac{1}{2}\triangle\Phi_1) + \frac{1}{2}\nabla\Phi_1.\nabla\Phi_2$ and $\frac{|\nabla\Phi_2|^2}{4} - \frac{1}{2}\triangle\Phi_2$ are nonnegative. One sees that the sublevel sets of $\frac{|\nabla\Phi|^2}{4} - \frac{1}{2}\triangle\Phi$ are included in the sublevel sets $\frac{|\nabla\Phi_2|^2}{4} - \frac{1}{2}\triangle\Phi_2$ an then are "thin at infinity" whence $-\triangle_\Phi$ generates a (holomorphic) compact C_0-semigroup in $L^1(\mathbb{R}^N; dx)$. ■

A classical result by D. Bakry and M. Emery (see e.g. [63] Théorème 3.1.29, p. 50) asserts that if Φ is uniformly strictly convex with $\int e^{-\Phi(x)}dx = 1$ then the probability measure $\mu(dx) = e^{-\Phi(x)}dx$ satisfies a logarithmic-Sobolev (or Gross) inequality and consequently (see e.g. [63] Proposition 3.1.8, p. 37) the spectral gap (or Poincaré) inequality holds. We complement this by the following result which does *not* depend on the integrability of $e^{-\Phi(x)}$:

Corollary 50 *Let Φ be uniformly strictly convex (i.e. $\exists\ m > 0,\ \Phi''(x) \geqslant mI\ \forall x \in \mathbb{R}^N$) such that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi$ is bounded below. Then $-\triangle_\Phi$ generates a (holomorphic) compact C_0-semigroup in $L^1(\mathbb{R}^N; dx)$.*

Proof. Let $\Phi''(x)$ be the Hessian of Φ at x. Let $\Phi_1(x) = \Phi(x) - \frac{m}{3}|x|^2$. Then $\Phi_1''(x)(h,h) = \Phi''(x)(h,h) - \frac{2m}{3}|h|^2 \geqslant \frac{m}{3}|h|^2$, i .e. $\Phi_1''(x) \geqslant \frac{m}{3}I$ so Φ_1 is uniformly strictly convex and consequently (see e.g. [63] p. 48) $x.\nabla\Phi_1(x) \geqslant \frac{m}{3}|x|^2 - b$ where b is a constant. Thus $\Phi(x) = \Phi_1(x) + \Phi_2(x)$ (where $\Phi_2(x) = \frac{m}{3}|x|^2$) with $\nabla\Phi_1(x).\nabla\Phi_2(x) = \frac{2m}{3}x.\nabla\Phi_1(x) \geqslant \frac{2m^2}{9}|x|^2 - \frac{2m}{3}b$. It follows that $\frac{|\nabla\Phi_1|^2}{4} - \frac{1}{2}\triangle\Phi_1$ is bounded from below since $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\triangle\Phi_2 = \frac{m^2}{9}|x|^2 - \frac{mN}{2}$ is. This ends the proof since $\frac{|\nabla\Phi_2(x)|^2}{4} - \frac{1}{2}\triangle\Phi_2 \to +\infty$ as $|x| \to \infty$. ■

We find in [27] systematic results on resolvent compactness or spectral gaps when Φ is a polynomial. In particular, if Φ is a sum of nonpositive monomials then $\triangle_\Phi$ is resolvent compact in $L^2(\mathbb{R}^N; dx)$ if and only if $\sum_{|\alpha|>0}|D_x^\alpha\Phi(x)| \to +\infty$ as $|x| \to +\infty$, see [27] Theorem 11.10 (ii), p. 120. We complement this by:

Proposition 51 *Let*

$$\Phi(x) = -\sum_{|\alpha|\leq C} c_\alpha x_1^{2\alpha_1}x_2^{2\alpha_2}...x_N^{2\alpha_N}, \quad (c_\alpha > 0) \tag{82}$$

where $\overline{\alpha}_i > 0\ \forall i$ for at least one multi-index $\overline{\alpha}$. Then $-\triangle_\Phi$ generates a (holomorphic) compact C_0-semigroup in $L^1(\mathbb{R}^N)$.

Proof. We have

$$\frac{\partial\Phi}{\partial x_j} = -\sum_{|\alpha|\leq C}(2\alpha_j c_\alpha)x_j^{2\alpha_j - 1}\Pi_{i\neq j}x_i^{2\alpha_i}$$

$$\frac{\partial^2\Phi}{\partial x_j^2} = -\sum_{|\alpha|\leq C}(2\alpha_j - 1)(2\alpha_j c_\alpha)x_j^{2\alpha_j-2}\Pi_{i\neq j}x_i^{2\alpha_i} \leq 0$$

so that $-\triangle\Phi \geqslant 0$. On the other hand

$$\begin{aligned}
|\nabla\Phi|^2 &= \sum_{j=1}^{N}\left[\sum_{|\alpha|\leq C}(2\alpha_j c_\alpha)x_j^{2\alpha_j-1}\Pi_{i\neq j}x_i^{2\alpha_i}\right]^2 \\
&\geqslant \sum_{j=1}^{N}\sum_{|\alpha|\leq C}(2\alpha_j c_\alpha)^2 x_j^{2(2\alpha_j-1)}\Pi_{i\neq j}x_i^{4\alpha_i} \\
&\geqslant \sum_{j=1}^{N}(2\overline{\alpha}_j c_{\overline{\alpha}})^2 x_j^{2(2\overline{\alpha}_j-1)}\Pi_{i\neq j}x_i^{4\overline{\alpha}_i}.
\end{aligned}$$

We observe that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi \geqslant 0$ and $\left\{x; \frac{1}{4}|\nabla\Phi(x)|^2 - \frac{1}{2}\triangle\Phi(x) \leq M\right\}$ is included in

$$\left\{x; x_j^{2(2\overline{\alpha}_j-1)}\Pi_{i\neq j}x_i^{4\overline{\alpha}_i} \leq \frac{4M}{(2\overline{\alpha}_j c_{\overline{\alpha}})^2}\right\}$$

for any j. It suffices to show that the latter set is thin at infinity. We may also restrict ourselves to positive coordinates. This set is defined by

$$x_j \leq \frac{M_j}{\Pi_{i\neq j}x_i^{\frac{2\overline{\alpha}_i}{(2\overline{\alpha}_j-1)}}}$$

where

$$M_j = \left[\frac{4M}{(2\overline{\alpha}_j c_{\overline{\alpha}})^2}\right]^{\frac{1}{2(2\overline{\alpha}_j-1)}}.$$

To fix the notations, suppose that $j = N$ and set

$$\beta_i := \frac{2\overline{\alpha}_i}{(2\overline{\alpha}_N - 1)},\ 1 \leq i \leq N-1.$$

Note first that if a_N is large enough then the interesection of a cube

$$C := \{x; a_i - 1 \leq x_i \leq a_i + 1;\ \forall i\}$$

with the set defined by $x_N \leq \frac{M_N}{\Pi_{i=1}^{N-1} x_i^{\beta_i}}$ is empty. On the other hand, it is true that the Lebesgue measure of this interesection is always less than

$$\begin{aligned} & M_N \int_{a_1-1}^{a_i+1} \frac{dx_1}{x_1^{\beta_1}} ... \int_{a_{N-1}-1}^{a_{N-1}+1} \frac{dx_{N-1}}{x_{N-1}^{\beta_{N-1}}} \\ = \; & M_N \left[\frac{1}{(1-\beta_1)} \left(\frac{1}{(a_1-1)^{\beta_1}} - \frac{1}{(a_1+1)^{\beta_1}} \right) \right] ... \\ & ... \left[\frac{1}{(1-\beta_{N-1})} \left(\frac{1}{(a_{N-1}-1)^{\beta_1}} - \frac{1}{(a_{N-1}+1)^{\beta_1}} \right) \right] \end{aligned}$$

when $\beta_i \neq 1$, otherwise replace the corresponding term by $\ln(\frac{(a_i+1)}{(a_i+1)})$. One sees that

$$M_N \int_{a_1-1}^{a_i+1} \frac{dx_1}{x_1^{\beta_1}} ... \int_{a_{N-1}-1}^{a_{N-1}+1} \frac{dx_{N-1}}{x_{N-1}^{\beta_{N-1}}} \to 0$$

if (at least) one coordinate a_i $(1 \leq i \leq N-1)$ tends to infinity. ∎

The case of nonnegative polynomials

$$\Phi(x) = \sum_{|\alpha| \leq C} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} ... x_N^{2\alpha_N}, \quad (c_\alpha > 0) \tag{83}$$

is much more involved even for homogeneous polynomials, see [27]. We restrict ourselves to the simplest "elliptic" case.

Proposition 52 *Let* $\Phi(x) = \sum_{|\alpha|=r} c_\alpha x_1^{2\alpha_1} x_2^{2\alpha_2} ... x_N^{2\alpha_N}$ $(c_\alpha > 0)$. *If* $\nabla\Phi(x) \neq 0$ *for* $x \neq 0$ *then* $-\triangle_\Phi$ *generates a (holomorphic) compact* C_0*-semigroup in* $L^1(\mathbb{R}^N)$.

Proof. It is known (see [27]) that $\frac{1}{4} |\nabla\Phi(x)|^2 - \frac{1}{2} \triangle\Phi(x) \to +\infty$ as $|x| \to \infty$; this is a consequence of the following facts: The compactness of the unit sphere S^{N-1} implies the existence of a constant $c > 0$ such that $|\nabla\Phi(x)| \geqslant c \; \forall x \in S^{N-1}$ and then $|\nabla\Phi(x)| \geqslant c\, |x|^{2r-1} \; \forall x \in \mathbb{R}^N$ since Φ is homogeneous of degree $2r$; on the other hand,

$$\triangle\Phi = \sum_{|\alpha|=r} \sum_{j=1}^{N} (2\alpha_j - 1)(2\alpha_j c_\alpha) x_j^{-2} x_1^{2\alpha_1} x_2^{2\alpha_2} ... x_N^{2\alpha_N}.$$

This ends the proof. ∎

Note that Proposition 52 covers e.g. the case

$$\Phi(x) = \sum_{i=1}^{N} c_i x_i^{2k} \quad (c_i > 0) \quad (k \geqslant 1).$$

Before giving one more example, let us come back to the model case (1) and observe that the sublevel sets of its potential $V(x_1, x_2) = x_1^2 x_2^2$, i.e.

$$\Omega_M = \left\{ (x_1, x_2);\ |x_2| \leq \frac{M}{|x_1|} \right\},$$

are thin at infinity. Indeed, it suffices to restrict ourselves to

$$\Omega_M^+ := \Omega_M \cap \{(x_1, x_2); x_1 > 0, x_2 > 0\} = \left\{ (x_1, x_2);\ x_2 \leq \frac{M}{x_1} \right\}$$

and to consider the case where we move the ball $B(z; 1)$ (centered at $z = (z_1, z_2)$ with $z_1 > 0$) by letting $z_1 \to +\infty$. The set $B(z; 1) \cap \Omega_M^+$ is included in $\{(x_1, x_2);\ z_1 - 1 \leq x_1 \leq z_1 + 1\} \cap \Omega_M^+$ whose Lebesgue measure is equal to

$$\int_{z_1-1}^{z_1+1} \frac{M}{x_1} dx_1 = M \ln\left(\frac{z_1+1}{z_1-1}\right) \to 0 \ \text{ as } z_1 \to +\infty.$$

We exploit this observation to deal with the weighted Laplacian corresponding to

$$\Phi(x_1, x_2) = x_1^2 x_2^2 + \varepsilon(x_1^2 + x_2^2) \ (\varepsilon > 0).$$

Indeed, it is known (see [27] Proposition 10.20, p. 111) that $\triangle_\Phi$ is resolvent compact in $L^2(\mathbb{R}^2)$ for *all* $\varepsilon > 0$. We can obtain a stronger conclusion for $\varepsilon \geqslant 1$. Indeed, one checks that

$$\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi = x_1^2 x_2^2(x_1^2 + x_2^2 + 2\varepsilon) + (\varepsilon^2 - 1)(x_1^2 + x_2^2) - 2\varepsilon \geqslant x_1^2 x_2^2(x_1^2 + x_2^2) - 2\varepsilon$$

so that, for $(x_1^2 + x_2^2) \geqslant 1$,

$$\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi \geqslant x_1^2 x_2^2 - 2\varepsilon$$

and then the above observation implies that $\triangle_\Phi$ generates a compact holomorphic semigroup in $L^1(\mathbb{R}^2)$. Note that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi$ is *not* bounded from below if $\varepsilon < 1$.

We end this section with an approach of *spectral gaps* for weighted Laplacians in terms of kernel estimates involving sublevel sets of

$$\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi$$

when the latter are *not* a priori thin at infinity. We restrict ourselves to the usual case

$$e^{-\Phi(x)} \in L^1(\mathbb{R}^N; dx). \tag{84}$$

Theorem 53 *Let Φ be a real C^2 function on $\mathbb{R}^N$ satisfying (84). Let $\frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi$ be nonnegative and let Ω_M be its sublevel sets. If*

$$\sup_{M>0} \lim_{C\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{\{x\in\Omega_M;\ |x|\geqslant C\}} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp(-\frac{|x-y|^2}{4t})dx < 1 \tag{85}$$

(for some $t>0$) then the C_0-semigroup generated by the weighted Laplacian on $L^2(\mathbb{R}^N;\mu(dx))$ has a spectral gap (but need not be compact).

Proof. If $e^{-\Phi(x)} \in L^1(\mathbb{R}^N;dx)$ then $\mu(dx)$ is finite and then the constant function 1 is an eigenfunction of $\triangle^\mu$ associated to the eigenvalue 0 which is then the spectral bound of $\triangle^\mu$. Then 0 is also the spectral bound of $\triangle - (\frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi)$ in $L^2(\mathbb{R}^N;dx)$ and also in $L^1(\mathbb{R}^N;dx)$ because the spectrum is the same in $L^2(\mathbb{R}^N;dx)$ and $L^1(\mathbb{R}^N;dx)$ (see e.g. [12]) whence $s(T_V)=0$ and we conclude by Theorem 37. ■

Remark 54 *One sees that (85) provides us with a sufficient condition (in terms of sublevel sets of $\frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi$) for the probability measure $\mu(dx) = Z^{-1}e^{-\Phi(x)}dx$ (where $Z=\int e^{-\Phi(x)}$) to satisfy the Poincaré inequality.*

8 On Witten Laplacians on 1-forms

Let Φ be a real C^2 function on $\mathbb{R}^N$ and let $\mu(dx) = e^{-\Phi(x)}dx$. Let

$$L^2(\mu) := L^2(\mathbb{R}^N,\ \mu(dx))$$

with scalar product $(.,.)_\mu$ and norm $\|\|_\mu$. The d-Complex in *weighted* L^2 spaces is given by

$$\Omega^0 \overset{d^{(0)}}{\to} \Omega^1 \overset{d^{(1)}}{\to} \Omega^2 \to \cdots\ \Omega^N \to 0$$

where $\Omega^p := \Omega^p(\mathbb{R}^N)$ ($p\leq N$) denotes the space of $L^2(\mu)$ p-forms (i.e. p-forms with coefficients in $L^2(\mu)$) equipped with its

$$L^2\left(\mathbb{R}^N,\ \mu;\ \wedge^p\mathbb{R}^N\right)$$

structure (Ω^0 is identified to $L^2(\mu)$). For the sake of simplicity, we still keep in Ω^p the notations $(.,.)_\mu$ and $\|\|_\mu$. Here

$$d^{(p)} : \Omega^p \to \Omega^{p+1}$$

is the restriction to Ω^p of the exterior differential d and is considered as an unbounded operator

$$L^2\left(\mathbb{R}^N,\ \mu;\ \wedge^p\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N,\ \mu;\ \wedge^{p+1}\mathbb{R}^N\right)$$

with domain

$$\left\{\omega \in \Omega^p;\ d\omega \in \Omega^{p+1}\right\}$$

where $d\omega$ is computed in the distributional sense. We denote by

$$d^{*(p)} : \Omega^{p+1} \to \Omega^p$$

the adjoint of $d^{(p)}$. The Laplacian $\triangle^{(p)}$ on Ω^p is then defined by

$$\triangle^{(p)} = d^{*(p)} \circ d^{(p)} + d^{(p-1)} \circ d^{*(p-1)} \quad (p \geqslant 1) \tag{86}$$

and

$$\triangle^{(0)} = d^{*(0)} \circ d^{(0)}.$$

Actually, the unbounded operator $\triangle^{(p)}$ is defined by means of its quadratic form

$$\left\| d^{(p)}\omega \right\|_\mu^2 + \left\| d^{*(p-1)}\omega \right\|_\mu^2, \quad \omega \in \Omega^p,$$

we refer to [70][32][26] for the details. It turns out that the Laplacian operator on *weighted* 0*-forms*

$$\triangle^{(0)} : L^2(\mu) \to L^2(\mu)$$

is unitarily equivalent to the following one

$$\triangle_\Phi^{(0)} = -\triangle + \frac{1}{4}\left|\nabla\Phi\right|^2 - \frac{1}{2}\triangle\Phi$$

on $L^2(\mathbb{R}^N,\ dx)$ while the Laplacian on *weighted* 1*-forms*

$$\triangle^{(1)} = d^{*(1)} \circ d^{(1)} + d^{(0)} \circ d^{*(0)}$$

on $L^2\left(\mathbb{R}^N,\ \mu;\ \wedge^1\mathbb{R}^N\right)$ is unitarily equivalent to the following one

$$\triangle_\Phi^{(1)} = \triangle_\Phi^{(0)} \otimes Id + Hess\Phi$$

on the unweighted space

$$L^2(\mathbb{R}^N,\ dx;\ \wedge^1\mathbb{R}^N)$$

where $Hess\Phi$ is the hessian of Φ; see [70][32][26].

We identify an 1-form to its coefficients and therefore the spaces

$$L^2(\mathbb{R}^N,\ dx;\ \wedge^1\mathbb{R}^N) = \left(L^2(\mathbb{R}^N,\ dx)\right)^N.$$

By construction, $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$ are *nonnegative* operators. Thus $\triangle_\Phi^{(1)}$ is a nonnegative unbounded operator on $\left(L^2(\mathbb{R}^N,\ dx)\right)^N$.

Spectral properties of Witten Laplacians $\triangle_\Phi^{(0)}$ on 0-forms have been considered in the previous section. Our aim now is to show the existence of *spectral connections* between $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$ (see e.g. [32] Theorem 1.3 for other kinds of connections). To this end, we recall first a basic functional analytic result related to Glazman's Lemma.

Theorem 55 *([56] Proposition 6.1.4, Corollaries 6.1.1 and 6.1.2, p. 72). Let A and B be two self-adjoint operators in a Hilbert space $\mathcal{H}$ such that*

$$(Au, u) \leq (Bu, u),\ u \in \mathcal{D}$$

where $\mathcal{D} \subset \mathcal{H}$ is a core for both A and B. Then:

(i) For any real λ, if $\sigma(A) \cap (-\infty, \lambda)$ is discrete (i.e. consists of isolated eigenvalues with finite multilpicities) then so si $\sigma(B) \cap (-\infty, \lambda)$.

(ii) If we denote by $\lambda_1^A \leq \lambda_2^A \leq \cdots \leq \lambda_k^A \leq \cdots$ and $\lambda_1^B \leq \lambda_2^B \leq \cdots \leq \lambda_k^B \leq \cdots$ their eigenvalues in $(-\infty, \lambda)$, numbered according to their multiplicities, then $\lambda_k^A \leq \lambda_k^B$.

If A is a bounded below self-adjoint operator then we define its essential lower spectral bound λ_{ess} as the supremum of the set

$$\{\lambda; \sigma(A) \cap (-\infty, \lambda) \ \text{ consists of isolated eigenvalues with finite multiplicity}\}$$

with the convention that $\lambda_{ess} = +\infty$ if the set is empty or equivalently if A is resolvent compact.

We give first spectral results under a *convexity* assumption on Φ.

Theorem 56 *Let Φ be a convex $\mathcal{C}^2$ function and let $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$ be the Laplacians defined above. Let λ_{ess}^0 and λ_{ess}^1 be respectively the essential lower spectral bounds of $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$. Then*

$$\lambda_{ess}^0 \leq \lambda_{ess}^1;$$

in particular, $\triangle_\Phi^{(1)}$ is resolvent compact if $\triangle_\Phi^{(0)}$ is. Let λ^0 and λ^1 be respectively the lower spectral bounds of $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$. If λ^0 is an isolated *eigenvalue of* $\triangle_\Phi^{(0)}$ *(i.e.* $\triangle_\Phi^{(0)}$ *has a spectral gap) and if the lowest eigenvalue* $\lambda_\Phi(x)$ *of* $Hess\Phi(x)$ *is not identically zero then*

$$\lambda^1 > \lambda^0.$$

Proof. Let $A = \triangle_\Phi^{(0)} \otimes Id$ and $B = \triangle_\Phi^{(1)}$. The convexity of Φ implies that $Hess\Phi$ is a *form-nonnegative* multiplication (matrix) operator so that $(A\omega,\omega) \leq (B\omega,\omega)$ for $\mathcal{C}_c^\infty$ 1-forms ω. Note that A is nothing but N copies of $\triangle_\Phi^{(0)}$ so that A has the same spectral strucure as $\triangle_\Phi^{(0)}$. In particular, the essential lower bound of $\triangle_\Phi^{(0)}$ coincides with that of A. Thus $\sigma(A) \cap (-\infty, \lambda_{ess}^0)$ is discrete and then, by Theorem 55, $\sigma(B) \cap (-\infty, \lambda_{ess}^0)$ is also discrete so that $\lambda_{ess}^0 \leq \lambda_{ess}^1$. If $\triangle_\Phi^{(0)}$ is resolvent compact then $\lambda_{ess}^0 = +\infty$ and then so is λ_{ess}^1 so $\triangle_\Phi^{(1)}$ is resolvent compact too.

To prove the last claim, note that

$$Hess\Phi \geqslant \lambda_\Phi(x) Id$$

implies

$$(\triangle_\Phi^{(0)} + \lambda_\Phi) \otimes Id \leq \triangle_\Phi^{(1)} \tag{87}$$

and then the spectral bottom of $(\triangle_\Phi^{(0)} + \lambda_\Phi) \otimes Id$ (or equivalently the spectral bottom $\widetilde{\lambda}^0$ of $\triangle_\Phi^{(0)} + \lambda_\Phi$) is less than or equal to that of $\triangle_\Phi^{(1)}$, i.e.

$$\widetilde{\lambda}^0 \leq \lambda^1.$$

It suffices to show that $\lambda^0 < \widetilde{\lambda}^0$. Note that $\lambda_\Phi \geqslant 0$ by the convexity of Φ and then $\triangle_\Phi^{(0)} \leq \triangle_\Phi^{(0)} + \lambda_\Phi$ implies the trivial inequality $\lambda^0 \leq \widetilde{\lambda}^0$. Suppose now that λ^0 is an isolated eigenvalue of $\triangle_\Phi^{(0)}$. Then there exists $\alpha > 0$ such that $\sigma(\triangle_\Phi^{(0)}) \cap \big[\lambda^0, \lambda^0 + \alpha\big)$ is discrete and then, by Theorem 55, $\sigma(\triangle_\Phi^{(0)} + \lambda_\Phi) \cap \big[\lambda^0, \lambda^0 + \alpha\big)$ is also discrete (possibly empty). Thus, if $\widetilde{\lambda}^0 \geqslant \lambda^0 + \alpha$ we are done. Otherwise, $\widetilde{\lambda}^0$ is an isolated eigenvalue of $\triangle_\Phi^{(0)} + \lambda_\Phi$; by a classical result this eigenvalue is simple and is associated to a normalized positive almost everywhere eigenfunction $\widetilde{f}$. By assumption, there exists also a normalized positive almost everywhere eigenfunction f associated to the eigenvalue λ^0 of $\triangle_\Phi^{(0)}$. The fact that $(f, \lambda_\Phi \widetilde{f}) > 0$ when $\lambda_\Phi(.)$ is not identically zero implies

$$\lambda^0(f,\widetilde{f}) = (\triangle_\Phi^{(0)} f, \widetilde{f}) = (f, \triangle_\Phi^{(0)} \widetilde{f}) < (f, \triangle_\Phi^{(0)} \widetilde{f} + \lambda_\Phi \widetilde{f}) = \widetilde{\lambda}^0 (f, \widetilde{f})$$

so that $\lambda^0 < \widetilde{\lambda}^0$. ■

Under the assumptions of the preceding theorem, if $\int e^{-\Phi(x)}dx = 1$ then $\lambda^0 = 0$ so $\lambda^1 > 0$ and consequently $\triangle_\Phi^{(1)}$ is invertible. This allows thus the formulation of the "exact" Helffer- Sjöstrand's covariance formula while Brascamp-Lieb's inequality

$$\int (f(x) - \langle f \rangle)(g(x) - \langle g \rangle) e^{-\Phi(x)} dx \leq \left((Hess\Phi)^{-1} df, dg \right)$$

is meaningful for strictly convex Φ only; see [32] for more information.

We *remove* now the convexity assumption on Φ.

Theorem 57 *Let Φ be a $\mathcal{C}^2$ function and let $\triangle_\Phi^{(0)}$ and $\triangle_\Phi^{(1)}$ be the Laplacians defined above. Let $\lambda_\Phi(x)$ be the lowest eigenvalue of $Hess\Phi(x)$. We assume that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi$ is bounded below. Then $\triangle_\Phi^{(1)}$ is resolvent compact provided that the sublevel sets of $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi$ are thin at infinity.*

Proof: It follows from (87) and Theorem 55, that $\triangle_\Phi^{(1)}$ is resolvent compact if $\triangle_\Phi^{(0)} + \lambda_\Phi$ is; the remainder is clear. ■

We show now how *spectral gaps* for Witten Laplacians on 1-forms occur when the sublevel sets of $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi$ are *not* thin at infinity. We still assume that assume that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi$ is bounded below; for simplicity, we assume that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi \geqslant 0$ (otherwise we "shift" the operator by adding a suitable constant). Let D^1 be the space of 1-form $\omega = \sum_{j=1}^N \omega_j dx_j$ with $\omega_j \in H^1(\mathbb{R}^N)$ and

$$\sum_{j=1}^N \int (\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi)|\omega_j(x)|^2 dx + \int (Hess\Phi(x)\omega(x), \omega(x))_{\mathbb{R}^N} dx < \infty.$$

The lower spectral bound of $\triangle_\Phi^{(1)}$ is given by

$$\begin{aligned}\lambda^1 \ : \ &= \inf_{\omega \in D^1, \|\omega\|_{L^2}=1} \sum_{j=1}^N \left[\int |\nabla\omega_j(x)|^2 dx + \int (\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi)|\omega_j(x)|^2 dx \right] \\ &+ \int (Hess\Phi(x)\omega(x), \omega(x))_{\mathbb{R}^N} dx\end{aligned}$$

while the lower spectral bound of $\triangle_\Phi^{(0)} + \lambda_\Phi$ is given by

$$\lambda^0 := \inf_{f \in D^0, \|f\|_{L^2}=1} \left[\int |\nabla f(x)|^2 dx + \int (\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi)|f(x)|^2 dx \right]$$

where

$$D^0 = \left\{ f \in H^1(\mathbb{R}^N);\ \int (\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi)\,|f(x)|^2\,dx < \infty \right\}.$$

Clearly $\lambda^0 \leq \lambda^1$.

Theorem 58 *Let Φ be a $\mathcal{C}^2$ function such that $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi \geqslant 0$. Let $\triangle_\Phi^{(1)}$ be the Laplacian defined above and let λ^1 be its lower spectral bound. We denote by Ω_M the sublevel sets of $\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi$. If*

$$\sup_{M>0}\lim_{C\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{\{x\in\Omega_M;\ |x|\geqslant C\}}\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{|x-y|^2}{4t})dx < e^{-\lambda^1 t} \tag{88}$$

(for some $t>0$) then $\triangle_\Phi^{(1)}$ has a spectral gap.

Proof. Let β_{ess}^0 be the essential lower spectral bound of $\triangle_\Phi^{(0)} + \lambda_\Phi$. Under (88), Theorem 37, with the heat semigroup $(U(t))_{t\geq 0}$ on $L^1(\mathbb{R}^N)$ and the potential

$$V = \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi,$$

and Remark 42 show that the essential type of the perturbed C_0-semigroup $(U_V(t))_{t\geq 0}$ in $L^1(\mathbb{R}^N)$ generated by

$$-(\triangle_\Phi^{(0)} + \lambda_\Phi) = \triangle - \left(\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\triangle\Phi + \lambda_\Phi\right) = \triangle - V$$

is *strictly* less $-\lambda^1$. On the other hand, the domination

$$U_V(t) \leq U(t)$$

shows that the kernel of $U_V(t)$ has a *Gaussian upper bound* and this implies that its (essential) spectrum is the same in all $L^p(\mathbb{R}^N)$ (see [12]). In particular, its essential type in $L^2(\mathbb{R}^N)$ is *strictly* less $-\lambda^1$ i.e. $-\beta_{ess}^0 < -\lambda^1$. Since $\sigma(\triangle_\Phi^{(0)} + \lambda_\Phi(x)) \cap (-\infty, \beta_{ess}^0)$ is discrete, or equivalently

$$\sigma((\triangle_\Phi^{(0)} + \lambda_\Phi(x)) \otimes Id) \cap (-\infty, \beta_{ess}^0) \text{ is discrete,}$$

then (87) and Theorem 55 show that $\sigma(\triangle_\Phi^{(1)}) \cap (-\infty, \beta_{ess}^0)$ is discrete. The fact that $\beta_{ess}^0 > \lambda^1$ shows that $\triangle_\Phi^{(1)}$ has a spectral gap. ∎

Remark 59 *An alternative approach to spectral theory of Witten Laplacians on 1-forms and Witten Laplacians on (0,1) forms is given in [50].*

9 Perturbation theory for indefinite potentials

This last section continues the general theory of Section 3 for general measure spaces

$$(\Omega; \mathcal{A}, \mu)$$

and deals with *indefinite* potentials

$$V = V_+ - V_-$$

(which are not a priori bounded from below) given as differences of nonnegative and finite almost everywhere functions (denoted by) V_+ and V_-. Note that V_+ and V_- need not be the positive and negative parts of V.

Let $(U(t))_{t \geqslant 0}$ be a substochastic C_0-semigroup in $L^1(\Omega; \mathcal{A}, \mu)$ with generator T. This section deals with spectral properties of

$$\text{“}T - V\text{”} = \text{“}T - V_+ + V_-\text{”}.$$

9.1 L^1 theory

We first define “$T - V_+ + V_-$”. Let V_+ satisfy (6)(8) and assume

$$V_- : D(T_{V_+}) \to L^1(\Omega; \mu) \text{ is } T_{V+}\text{-bounded} \tag{89}$$

with

$$\lim_{\lambda \to +\infty} r_\sigma \left[V_-(\lambda - T_{V_+})^{-1} \right] < 1. \tag{90}$$

Then Desch's theorem [14] (see e.g. [4] Chapter 5 or [42] Chapters 7 and 8) shows that

$$T_{V+} + V_- : D(T_{V_+}) \to L^1(\Omega; \mu)$$

generates a positive C_0-semigroup $\left(e^{t(T_{V_+} + V_-)}\right)_{t \geq 0}$ on $L^1(\Omega; \mu)$.

The spectral properties (full discreteness or spectral gaps) of T_{V_+} and $\left(e^{tT_{V_+}}\right)_{t \geq 0}$ are dealt with in details in Section 3, Section 5 and Section 6. In the present section, we show how these spectral properties are *inherited* by $T_{V+} + V_-$ and $\left(e^{t(T_{V_+} + V_-)}\right)_{t \geq 0}$. This perturbed C_0-semigroup is given by a Dyson-Phillips series

$$e^{t(T_{V_+} + V_-)} = \sum_{k=0}^{\infty} U_k(t) \tag{91}$$

where $U_0(t) = e^{tT_{V+}}$ and

$$U_k(t)\varphi = \int_0^t U_{k-1}(s)V_- U_0(t-s)\varphi ds \quad (k \geq 1) \tag{92}$$

where the operators $U_k(t)$ defined (by induction) first on $D(T_{V+})$ extend uniquely as bounded operators on $L^1(\Omega;\mu)$ and the series (91) converges in *operator norm* and *uniformly in bounded* t; see e.g [42] Chapters 7 and 8 for the details. By renorming the space $L^1(\Omega;\mu)$ by an equivalent norm $\|\|\|$, *additive on the positive cone*, without loss of generality we can replace (90) by

$$\lim_{\lambda\to+\infty} \left\|V_-(\lambda - T_{V_+})^{-1}\right\| < 1, \tag{93}$$

(see [42] Lemma 8.3, p. 189). We *fix* λ large enough such that

$$\left\|V_-(\lambda - T_{V_+})^{-1}\right\| < 1.$$

By shifting T_{V_+} by $-\lambda I$ (i.e. we replace T_{V_+} by $T_{V_+} - \lambda$) we can assume without loss of generality that $s(T_{V_+}) < 0$ and

$$\left\|V_-(0 - T_{V_+})^{-1}\right\| < 1.$$

Let

$$\mathcal{X}_{\bar{t}} = C\left([0,+\infty),\ \mathcal{L}(L^1(\Omega,\mu))\right)$$

denote the Banach space of *strongly continuous* $\mathcal{L}(L^1(\Omega,\mu))$-valued functions equipped with sup-norm

$$\|Z\|_\infty = \sup_{t\in[0,+\infty)} \|Z(t)\|_{\mathcal{L}(L^1(\Omega,\mu))}$$

and define the linear operator on $\mathcal{X}_{\bar{t}}$

$$\mathcal{O} : \mathcal{X}_{\bar{t}} \ni Z \to \int_0^t Z(s)V_- U_0(t-s)ds \in \mathcal{X}_{\bar{t}}.$$

Let us estimate the norm of $\mathcal{O}Z$. Note that for $\varphi \in D(T_{V+})$

$$\begin{aligned}
\|\mathcal{O}Z(t)\varphi\| &\leq \int_0^t \|Z(s)V_- U_0(t-s)\varphi\| ds \leq \|Z\|_\infty \int_0^t \|V_- U_0(t-s)\varphi\| ds \\
&\leq \|Z\|_\infty \int_0^t \|V_- U_0(t-s)\,|\varphi|\|\, ds.
\end{aligned}$$

By the *additivity of the norm* on the positive cone,

$$\begin{aligned}
\int_0^t \|V_- U_0(t-s)\,|\varphi|\|\,ds &= \left\|\int_0^t V_- U_0(s)\,|\varphi|\,ds\right\| \le \left\|\int_0^{+\infty} V_- U_0(s)\,|\varphi|\,ds\right\| \\
&= \left\|V_- \int_0^{+\infty} U_0(s)\,|\varphi|\,ds\right\| = \left\|V_-(0-T_{V_+})^{-1}\,|\varphi|\right\| \\
&\le \left\|V_-(0-T_{V_+})^{-1}\right\|_{\mathcal{L}(L^1(\Omega,\mu))} \||\varphi|\| \\
&= \left\|V_-(0-T_{V_+})^{-1}\right\|_{\mathcal{L}(L^1(\Omega,\mu))} \|\varphi\|
\end{aligned}$$

so

$$\|\mathcal{O}Z(t)\varphi\| \le \|Z\|_\infty \left\|V_-(0-T_{V_+})^{-1}\right\|_{\mathcal{L}(L^1(\Omega,\mu))} \|\varphi\| \ \ \forall t \ge 0$$

and, by density, this estimate remains true for all $\varphi \in L^1(\Omega,\mu)$ so

$$\|\mathcal{O}Z\|_\infty \le \left\|V_-(0-T_{V_+})^{-1}\right\|_{\mathcal{L}(L^1(\Omega,\mu))} \|Z\|_\infty$$

and

$$\|\mathcal{O}\|_{\mathcal{L}(\mathcal{X}_{\bar{t}})} \le \left\|V_-(0-T_{V_+})^{-1}\right\|_{\mathcal{L}(L^1(\Omega,\mu))} < 1.$$

Thus V_- is a *Miyadera-Voigt* perturbation of T_{V_+} according to the terminology in [38].

We are ready to show:

Theorem 60 *Let $(U(t))_{t\geqslant 0}$ be substochastic C_0-semigroup in $L^1(\Omega;\mathcal{A},\mu)$ with generator T and let V_+ satisfy (6)(8). We assume that (89) (90) are satisfied. Then:*

(i) If T_{V_+} is resolvent compact then so is $T_{V_+}+V_-$.

(ii) If $\left(e^{tT_{V_+}}\right)_{t\ge 0}$ is compact then so is $\left(e^{t(T_{V_+}+V_-)}\right)_{t\ge 0}$.

Proof. Let T_{V_+} be resolvent compact. The perturbed resolvent for λ large enough

$$(\lambda - T_{V_+} - V_-)^{-1} = (\lambda - T_{V_+})^{-1} \sum_{i=0}^{+\infty} (V_-(\lambda - T_{V_+})^{-1})^i \tag{94}$$

shows that $T_{V+}+V_-$ is also resolvent compact.

Let $\left(e^{tT_{V_+}}\right)_{t\ge 0}$ be compact. Then T_{V_+} is resolvent compact (see [57] Theorem 3.3, p. 48) and consequently, by (i), so is $T_{V+}+V_-$. On the other hand, $\left(e^{tT_{V_+}}\right)_{t\ge 0}$ is also operator norm continuous (see [57] Theorem 3.3).

Since V_- is a *Miyadera-Voigt* perturbation then the operator norm continuity of $\left(e^{tT_{V+}}\right)_{t\geq 0}$ is inherited by $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$, (see [38], Theorem 9). The operator norm continuity of $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$ and the resolvent compactness of $T_{V+}+V_-$ imply (see [57] Theorem 3.3) that $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$ is compact. ■

We deal now with spectral gaps for generators.

Theorem 61 *Let $(U(t))_{t\geqslant 0}$ be a substochastic C_0-semigroup in $L^1(\Omega;\mathcal{A},\mu)$ with generator T and let V_+ satisfy (6)(8). We assume that*

$$V_- \text{ is } T_{V+}\text{-weakly compact} \tag{95}$$

i.e. $V_-(\lambda - T_{V_+})^{-1}$ is weakly compact. Then (90) is satisfied and

$$s_{ess}(T_{V+}+V_-) = s_{ess}(T_{V+}). \tag{96}$$

In particular

$$s(T_{V+}+V_-) - s_{ess}(T_{V+}+V_-) > 0$$

if

$$s(T_{V+}) - s_{ess}(T_{V+}) > 0.$$

Proof. It is known (see [44]) that (95) implies that

$$\lim_{\lambda\to+\infty} r_\sigma\left[V_-(\lambda - T_{V_+})^{-1}\right] = 0$$

so that (90) is satisfied. On the other hand (94) shows that

$$(\lambda - T_{V_+} - V_-)^{-1} - (\lambda - T_{V_+})^{-1} = (\lambda - T_{V_+})^{-1}\sum_{i=1}^{+\infty}(V_-(\lambda - T_{V_+})^{-1})^i$$

is weakly compact. It follows that $(\lambda - T_{V_+} - V_-)^{-1}$ and $(\lambda - T_{V_+})^{-1}$ have the same essential spectrum (see [36], Proposition 2.c.10, p. 79) so $T_{V_+}+V_-$ and T_{V_+} share the same essential spectrum and consequently (96) is satisfied. We note that $s(T_{V+}+V_-) \geq s(T_{V+})$ because

$$(\lambda - T_{V_+} - V_-)^{-1} \geq (\lambda - T_{V_+})^{-1}$$

so that

$$s(T_{V+}+V_-) - s_{ess}(T_{V+}+V_-) \geq s(T_{V+}) - s_{ess}(T_{V+})$$

and this ends the proof. ■

We consider now spectral gaps for C_0-semigroups.

Theorem 62 *Let $(U(t))_{t\geqslant 0}$ be a substochastic C_0-semigroup in $L^1(\Omega;\mathcal{A},\mu)$ with generator T and let V_+ satisfy (6)(8). Let (95) be satisfied. We assume that*

$$\left(e^{tT_{V_+}}\right)_{t\geq 0} \text{ is operator norm-continuous} \tag{97}$$

Then $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$ and $\left(e^{tT_{V_+}}\right)_{t\geq 0}$ share the same essential spectrum and consequently the same essential type. In particular, $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$ has a spectral gap if $\left(e^{tT_{V_+}}\right)_{t\geq 0}$ has.

Proof. We have seen in the proof of Theorem 60 that $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$ is also operator norm-continuous. We start from

$$\int_0^{+\infty}\left(e^{s(T_{V_+}+V_-)}-e^{sT_{V_+}}\right)ds=(\lambda-T_{V_+})^{-1}\sum_{i=1}^{+\infty}(V_-(\lambda-T_{V_+})^{-1})^i$$

so that (for any $t>0$ and $\varepsilon>0$) the domination

$$(\lambda-T_{V_+})^{-1}\sum_{i=1}^{+\infty}(V_-(\lambda-T_{V_+})^{-1})^i\geq\int_t^{t+\varepsilon}\left(e^{s(T_{V_+}+V_-)}-e^{sT_{V_+}}\right)ds$$

shows that

$$\int_t^{t+\varepsilon}\left(e^{s(T_{V_+}+V_-)}-e^{sT_{V_+}}\right)ds \text{ is weakly compact}$$

ans then so is

$$e^{t(T_{V_+}+V_-)}-e^{tT_{V_+}}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left(e^{s(T_{V_+}+V_-)}-e^{sT_{V_+}}\right)ds\quad(t>0)$$

because the limit holds in operator norm since both C_0-semigroups are operator norm-continuous. The stability of the essential spectrum by a weakly compact perturbation (see [36], Proposition 2.c.10, p. 79) shows the first claim. The second claim follows from the fact that $s(T_{V+}+V_-)\geq s(T_{V+})$ and that these spectral bounds of the generators coincide with the types of the corresponding C_0-semigroups. ■

Note that if V_- is T-weakly compact, i.e. if $V_-(\lambda-T)^{-1}$ is weakly compact, (and therefore T-bounded) then

$$V_-(\lambda-T_{V_+})^{-1}\leq V_-(\lambda-T)^{-1} \tag{98}$$

shows that (95) is satisfied regardless of V_+. We put aside this particular case and give a sufficient condition insuring the key condition (95) which relies on a suitable *"competition"* between the components V_+ and V_- of the potential V.

Proposition 63 *Let $(U(t))_{t\geqslant 0}$ be a substochastic C_0-semigroup in $L^1(\Omega;\mathcal{A},\mu)$ with generator T and let V_+ satisfy (6)(8). Let $\Omega_M := \{y; V_+(y) \leq M\}$ be the sublevel sets of V_+. We assume that*

$$\frac{V_-}{V_+} \text{ is bounded and } \sup_{x\in\Omega_M^c} \frac{V_-(x)}{V_+(x)} \to 0 \text{ as } M \to +\infty. \tag{99}$$

If for all M

$$1_{\Omega_M} V_-(\lambda - T_{V_+})^{-1} \text{ is weakly compact} \tag{100}$$

then (95) is satisfied.

Proof. By assumption, there exists $c > 0$ such that

$$V_-(x) \leq cV_+(x) \text{ on } \Omega.$$

Note that

$$V_-(\lambda - T_{V_+})^{-1} = \frac{V_-}{V_+} V_+(\lambda - T_{V_+})^{-1}$$

shows that V_- is T_{V_+}-bounded since V_+ is T_{V_+}-bounded (Lemma 1). We have

$$\begin{aligned} V_-(\lambda - T_{V_+})^{-1} &= 1_{\Omega_M} V_-(\lambda - T_{V_+})^{-1} \\ &\quad + 1_{\Omega_M^c} V_-(\lambda - T_{V_+})^{-1}. \end{aligned}$$

By (100) $1_{\Omega_M} V_-(\lambda - T_{V_+})^{-1}$ is weakly compact for any $M > 0$. On the other hand

$$\begin{aligned} 1_{\Omega_M^c} V_-(\lambda - T_{V_+})^{-1} &= 1_{\Omega_M^c} \frac{V_-}{V_+} V_+(\lambda - T_{V_+})^{-1} \\ &\leq \left(\sup_{x\in\Omega_M^c} \frac{V_-(x)}{V_+(x)} \right) V_+(\lambda - T_{V_+})^{-1} \end{aligned}$$

goes to zero in norm as $M \to \infty$ by (99) since V_+ is T_{V_+}-bounded. ■

Corollary 64 *We note that (98) implies that the generalized Kato class of $(U(t))_{t\geqslant 0}$ is included in the generalized Kato class of $\left(U_{V_+}(t)\right)_{t\geq 0}$. See [45] for some results on generalized Kato class potentials for convolution C_0-semigroups $(U(t))_{t\geqslant 0}$ on $L^1(\mathbb{R}^N)$ (with generator T), in particular for T-weakly compact potentials V_-.*

9.2 L^p theory

Let $(U(t))_{t\geqslant 0}$ be a sub-Markov C_0-semigroup with generator T in $L^1(\Omega,\mu)$ (i.e. acts in all L^p spaces as a positive contraction C_0-semigroup). We denote it by $(U^p(t))_{t\geqslant 0}$ when acting on $L^p(\Omega,\mu)$ and denote its generator by T^p (so $T^1 = T$). We denote by $\left(U^p_{V_+}(t)\right)_{t\geq 0}$ the perturbed C_0-semigroup (for the potential V_+) and by $T^p_{V_+}$ its generator. Under (90) one shows that the C_0-semigroup $\left(e^{t(T_{V_+}+V_-)}\right)_{t\geq 0}$ on $L^1(\Omega,\mu)$, with generator

$$T_{V_+} + V_- : D(T_{V_+}) \to L^1(\Omega,\mu),$$

interpolates on all $L^p(\Omega,\mu)$ $(1 \leq p < \infty)$ providing positive strongly continuous semigroups $(W_p(t))_{t\geq 0} = \left(e^{tA_p}\right)_{t\geq 0}$ in $L^p(\Omega,\mu)$ with generators A_p where $A_1 = T_{V_+} + V_-$; (this is done in [45] for convolution C_0-semigroups but the ideas can be adapted easily to this general context). We point out that V_- is *not* a priori $T^p_{V_+}$-bounded for $p > 1$ and, as far as we know, there is no simple characterisation of the domain of A_p. However, if $(U(t))_{t\geqslant 0}$ is symmetric then V_- is *form-bounded* with respect to $-T^2_{V_+}$ with relative form-bound less than or equal to

$$\lim_{\lambda\to+\infty} r_\sigma\left[V_-(\lambda - T_{V_+})^{-1}\right]$$

and A_2 is given by

$$-A_2 = (-T^2_{V_+}) \dotplus (-V_-) \quad \text{(form-sum)}, \tag{101}$$

(see [45][46]).

Theorem 65 *Let $(U(t))_{t\geqslant 0}$ be a sub-Markov C_0-semigroup and let V_+ satisfy (6)(8). Let (90 be satisfied. If T_{V_+} is resolvent compact on $L^1(\Omega,\mu)$ then A_p is resolvent compact too. In the symmetric case, $(W_p(t))_{t\geq 0}$ is a compact C_0-semigroup on $L^p(\Omega,\mu)$ for all $p > 1$.*

Proof. By Theorem 60, $T_{V_+} + V_-$ is resolvent compact in $L^1(\Omega,\mu)$. By interpolation, A_p is resolvent compact too in $L^p(\Omega,\mu)$ for all $p > 1$. Since $(W_2(t))_{t\geq 0} = \left(e^{tA_2}\right)_{t\geq 0}$ is self-adjoint then it is operator norm continuous so that, by interpolation, $\left(e^{tA_p}\right)_{t\geq 0}$ (for $p > 1$) are operator norm continuous too. Finally $\left(e^{tA_p}\right)_{t\geq 0}$ is compact (see [57] Theorem 3.3). ■

Remark 66 *Let $(U(t))_{t\geqslant 0}$ be subordinated to the heat C_0-semigroup on $L^1(R^N)$ and let V_+ satisfy (3) (or equivalently let the sublevel sets of V_+*

be thin at infinity). Then T_{V_+} is resolvent compact on $L^1(R^N)$ by Corollary 28 and Theorem 65 implies that A_2 (as given by (101)) has a discrete spectrum; (see [7] for a result in this direction when $(U(t))_{t\geqslant 0}$ is the heat semigroup).

We end this section with:

Theorem 67 *Let $(U(t))_{t\geqslant 0}$ be a sub-Markov C_0-semigroup on $L^1(\Omega;\mathcal{A},\mu)$ with generator T and let V_+ satisfy (6)(8). Let (95) be satisfied. We assume that $(U(t))_{t\geqslant 0}$ is operator norm-continuous. Then $\left(e^{tA_p}\right)_{t\geq 0}$ and $\left(e^{tT^p_{V_+}}\right)_{t\geq 0}$ have the same essential type. In particular, $\left(e^{tA_p}\right)_{t\geq 0}$ has a spectral gap if $\left(e^{tT^p_{V_+}}\right)_{t\geq 0}$ has.*

Proof. Note first that (95) implies that

$$\lim_{\lambda\to+\infty} r_\sigma\left[V_-(\lambda - T_{V_+})^{-1}\right] = 0$$

(see [44]). We know that $e^{t(T_{V_+}+V_-)} - e^{tT_{V_+}}$ is weakly compact in $L^1(\Omega,\mu)$ for $t > 0$ (see the proof of Theorem 62) and then so is

$$(\alpha - e^{tT_{V_+}})^{-1}\left(e^{t(T_{V_+}+V_-)} - e^{tT_{V_+}}\right)$$

(for large $|\alpha|$). It follows that $\left[(\alpha - e^{tT_{V_+}})^{-1}\left(e^{t(T_{V_+}+V_-)} - e^{tT_{V_+}}\right)\right]^2$ is compact in $L^1(\Omega,\mu)$ (see e.g. [1] Corollary 5.88, p. 344) and consequently, by interpolation,

$$\left[(\alpha - e^{tT^p_{1V_+}})^{-1}\left(e^{tA_p} - e^{tT^p_{1V_+}}\right)\right]^2$$

is compact on $L^p(\Omega,\mu)$ for all $p > 1$. Finally, the analytic Fredholm alternative shows that e^{tA_p} and $e^{tT_{pV_+}}$ have the same essential radius (see e.g. [48] Corollary 7, p. 358) and consequently the same essential type. ■

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